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Some Identities Associated with a Discriminant

By G. N. Watson

(*Received 24th November* 1953.)

1. *Introduction.*

This paper has to deal with some topics suggested by a theorem which is notorious in the theory of quadratic forms, namely that the roots of the characteristic equation of any real quadratic form are all real. For a proof of this theorem, the reader may consult Ferrar (**1**), p. 146.

Take the quadratic form to be one in three variables, say

$$ax^2 + by^2 + cz^2 + 2uyz + 2vzx + 2wxy,$$

with a, b, c, u, v, w all real, so that the characteristic equation is the cubic

$$\begin{vmatrix} a-\lambda & w & v \\ w & b-\lambda & u \\ v & u & c-\lambda \end{vmatrix} = 0.$$

A change of the unknown from λ to μ by the substitution

$$3\lambda = 3\mu + \Sigma a$$

deprives the cubic of its second term; we then write

$$3a' = 3a - \Sigma a, \qquad 3b' = 3b - \Sigma a, \qquad 3c' = 3c - \Sigma a,$$

and the characteristic equation becomes

$$\begin{vmatrix} a'-\mu & w & v \\ w & b'-\mu & u \\ v & u & c'-\mu \end{vmatrix} = 0,$$

wherein $$a' + b' + c' = 0.$$

This is of the form $\mu^3 + 3H\mu + G = 0$, with

$$\begin{aligned} 9H &\equiv 3\Sigma b'c' - 3\Sigma u^2 \\ &\equiv \Sigma bc - \Sigma a^2 - 3\Sigma u^2 \end{aligned}$$

and

$$\begin{aligned} 27G &\equiv -27\,(a'b'c' + 2uvw - \Sigma a'u^2) \\ &\equiv -\Pi\,(2a-b-c) - 54uvw + 9\Sigma\,(2a-b-c)u^2, \end{aligned}$$

the summations and product running over a, b, c, or over u, v, w, or over a, b, c and u, v, w jointly.

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the summations and product running over a, b, c, or over $u, v\ w$, or over a, b, c and u, v, w jointly.

Now a necessary and sufficient condition for the reality of the three roots of a cubic equation with real coefficients is that the product of the squared differences of the roots should be positive in the loose sense. For the cubics in λ and μ this product is equal to $-27(4H^3+G^2)$; we thus obtain the following theorem, due to Kummer (2):

THEOREM 1. *Let*

$$(1)\quad 27F(a,b,c;\,u,v,w) \equiv 4(\Sigma a^2-\Sigma bc+3\Sigma u^2)^3 - \{\Pi(2a-b-c)+54uvw-9\Sigma(2a-b-c)u^2\}^2;$$

then

$$(2)\qquad F(a,b,c;\,u,v,w)\geqq 0$$

for all real values of the numbers a, b, c, u, v, w.

It was remarked by Kummer that this theorem suggested that it might be possible to express $F(a,b,c;\,u,v,w)$ as a sum of squares, and he proceeded to state such an expression after giving the corresponding formula for the special case $F(0,0,0;\,u,v,w)$; this expression is given by formula (14) *infra*. He gave no indication of how he discovered these results. Some forty years later Tannery (3) gave a derivation of the formulae by the methods of analytical geometry; although his work does not involve any complicated algebra, it seemed to me to be very obscure, and I decided that it might be worth while to derive the formulae in as straightforward and natural a manner as possible; apart from quoting one theorem about invariance, I use nothing more than the elementary algebra of complex numbers.

It was noticed by Kummer that a, b, c enter F only by their differences; and we can abbreviate the subsequent work substantially by writing

$$(3)\qquad b-c=\alpha,\qquad c-a=\beta,\qquad a-b=\gamma,$$

so that

$$(4)\qquad \alpha+\beta+\gamma=0.$$

With this notation, the expansion of $F(a,b,c;\,u,v,w)$ in ascending powers of u, v, w assumes the form

$$\begin{aligned}(5)\qquad & F(a,b,c;\,u,v,w)\\ &\equiv \alpha^2\beta^2\gamma^2+\Sigma(4\beta^2\gamma^2+2\alpha^2\beta\gamma)u^2+4uvw\,\Pi(\beta-\gamma)\\ &\quad+\Sigma(4\alpha^2+8\beta\gamma)u^4+\Sigma(20\alpha^2-2\beta\gamma)v^2w^2\\ &\quad-36uvw\,\Sigma u^2(\beta-\gamma)+4(\Sigma u^2)^3-108u^2v^2w^2.\end{aligned}$$

This result shows incidentally that, in the special case in which a, b, c are all equal, Theorem 1 reduces to the theorem of the arithmetic and geometric means for u^2, v^2, w^2.

2. *Some special cases of the representation of* $F(a, b, c; u, v, w)$.

Before we attack the general problem of representing the expression $F(a, b, c; u, v, w)$ as a sum of squares, we consider special cases in which at least one of the numbers u, v, w is zero. First, when at least two of them are zero (say $u = v = 0$), we have the almost trivial formula

$$F(a, b, c; 0, 0, w) \equiv (\gamma^2 + 4w^2)(a\beta + w^2)^2. \tag{6}$$

Next, let only one of the three numbers be zero (say $u = 0$, $v \neq 0$, $w \neq 0$). We replace the variables α, β, γ by new variables $\alpha_1, \beta_1, \gamma_1$ defined by the formulæ

$$\alpha_1 = \frac{\alpha}{\sqrt{(v^2 + w^2)}}, \qquad \beta_1 = \frac{\beta}{\sqrt{(v^2 + w^2)}}, \qquad \gamma_1 = \frac{\gamma}{\sqrt{(v^2 + w^2)}}.$$

This change transforms $F(a, b, c; 0, v, w)/(v^2 + w^2)$ into a homogeneous quadratic function of v^2 and w^2, the coefficients in the quadratic being functions of $\alpha_1, \beta_1, \gamma_1$. In fact, some straightforward simplifications show that

$$\begin{aligned} F(a, b, c; 0, v, w)/(v^2 + w^2) &\equiv v^4(\beta_1^2 + 4)(\alpha_1\gamma_1 + 1)^2 + w^4(\gamma_1^2 + 4)(\alpha_1\beta_1 + 1)^2 \\ &\quad + v^2w^2\{2\alpha_1^2\beta_1^2\gamma_1^2 + \alpha_1^2(4\beta_1^2 + 4\gamma_1^2 - 2\beta_1\gamma_1) \\ &\qquad + (19\alpha_1^2 + \beta_1^2 + \gamma_1^2) + 8\}. \end{aligned}$$

An inspection of the expression on the right shows that the whole of the terms in v^4 and w^4, together with a great part of the term in v^2w^2, is accounted for by the sum

$$\{v^2\beta_1(\alpha_1\gamma_1 + 1) + w^2\gamma_1(\alpha_1\beta_1 + 1)\}^2 + 4\{v^2(\alpha_1\gamma_1 + 1) + w^2(\alpha_1\beta_1 + 1)\}^2;$$

in fact, when we subtract this sum from the expression in question and use (4) freely to reduce the result, we are left with

$$v^2w^2\{4\alpha_1^2(\beta_1 - \gamma_1)^2 + 27\alpha_1^2 + (\beta_1 - \gamma_1)^2\},$$

and, by good fortune, this is a sum of squares. Hence we have

$$\begin{aligned} F(a, b, c; 0, v, w)/(v^2 + w^2) &\equiv \{(v^2 + w^2)\alpha_1\beta_1\gamma_1 + v^2\beta_1 + w^2\gamma_1\}^2 \\ &\quad + 4\{v^2\alpha_1\gamma_1 + w^2\alpha_1\beta_1 + v^2 + w^2\}^2 \\ &\quad + 4v^2w^2\alpha_1^2(\beta_1 - \gamma_1)^2 + v^2w^2\{27\alpha_1^2 + (\beta_1 - \gamma_1)^2\}. \end{aligned}$$

When we return to the original notation, this result becomes

$$(7)\quad F(a,b,c;\ 0,v,w).(v^2+w^2)$$
$$\equiv (v^2+w^2)(a\beta\gamma+v^2\beta+w^2\gamma)^2$$
$$+4\{v^2a\gamma+w^2a\beta+(v^2+w^2)^2\}^2+4v^2w^2a^2(\beta-\gamma)^2$$
$$+v^2w^2(v^2+w^2)\{27a^2+(\beta-\gamma)^2\}.$$

This result can be simplified by rearranging the second and third terms on the right in such a way that it becomes possible to cancel a factor v^2+w^2. To effect the rearrangement, observe that

$$v^2a\gamma+w^2a\beta+(v^2+w^2)^2\pm ivwa(\beta-\gamma)$$
$$\equiv (v\mp iw)\{v(a\gamma+v^2+w^2)\pm iw(a\beta+v^2+w^2)\},$$

where either all the upper signs or else all the lower signs are to be taken throughout the ambiguities. By means of this pair of formulæ we derive the following theorem from (7).

THEOREM 2. *The special function $F(a,b,c;\ 0,v,w)$ can be represented as a sum of squares by the formula*

$$(8)\quad F(a,b,c;\ 0,v,w)\equiv(a\beta\gamma+v^2\beta+w^2\gamma)^2$$
$$+4v^2(a\gamma+v^2+w^2)^2$$
$$+4w^2(a\beta+v^2+w^2)^2$$
$$+v^2w^2\{27a^2+(\beta-\gamma)^2\}.$$

It was assumed that v^2+w^2 is not zero in the course of proving Theorem 2, but it is evident that the theorem remains valid when this limitation is removed.

3. *Representations of the general function $F(a,b,c;\ u,v,w)$.*

Now that we have disposed of various special cases, we direct our attention to the general function $F(a,b,c;\ u,v,w)$ in which it is not assumed that uvw is zero. We derive identities from Theorem 2 by a process which is quite straightforward but rather tedious.

It is well known (Ferrar, **1**, p. 194) that the coefficients in the characteristic equation of a quadratic form are invariant under normal orthogonal transformations of the form; we take a special transformation of this type, namely

$$x=X,\qquad y=Y\cos\theta-Z\sin\theta,\qquad z=Y\sin\theta+Z\cos\theta,$$

with a value of θ which will be chosen presently; the original form

$$ax^2+by^2+cz^2+2uyz+2vzx+2wxy$$

is thereby converted into (say)

$$AX^2 + BY^2 + CZ^2 + 2UYZ + 2VZX + 2WXY,$$

with

$$(9)\quad \begin{cases} A = a, \quad U = \frac{1}{2}(c-b)\sin 2\theta + u\cos 2\theta, \\ B = b\cos^2\theta + c\sin^2\theta + 2u\sin\theta\cos\theta, \\ C = b\sin^2\theta + c\cos^2\theta - 2u\sin\theta\cos\theta, \\ V = v\cos\theta - w\sin\theta, \; W = v\sin\theta + w\cos\theta. \end{cases}$$

Since the product of the squared differences of the roots of a cubic equation is a rational function of the coefficients in the equation, the theorem on invariance which has just been quoted shows that

$$(10)\qquad F(a, b, c;\ u, v, w) \equiv F(A, B, C;\ U, V, W);$$

this result, of course, can be proved by a lengthy piece of elementary algebra, as a direct consequence of equations (9), without any appeal to theorems on invariance.

We now choose θ so that U vanishes, that is to say we take

$$(11)\qquad \cos 2\theta = \frac{a}{R}, \qquad \sin 2\theta = \frac{2u}{R},$$

where R denotes $\pm\sqrt{(a^2 + 4u^2)}$; it does not matter whether the positive or the negative value is assigned to R so long as one value is adopted throughout the work. With a value of θ satisfying (11) we have

$$\begin{aligned}(12)\quad F(a, b, c;\ u, v, w) &\equiv F(A, B, C;\ 0, V, W) \\ &\equiv \{(B-C)(C-A)(A-B) + V^2(C-A) + W^2(A-B)\}^2 \\ &\quad + 4V^2\{(B-C)(A-B) + V^2 + W^2\}^2 \\ &\quad + 4W^2\{(B-C)(C-A) + V^2 + W^2\}^2 \\ &\quad + V^2W^2(B+C-2A)^2 + 27V^2W^2(B-C)^2.\end{aligned}$$

We now have to express the right-hand side of (12) as a sum of squares of functions of $\alpha, \beta, \gamma, u, v, w$. As a preliminary we give the following results, which are easily obtained from (9):

$$\begin{aligned} B - C &= \alpha\cos 2\theta + 2u\sin 2\theta = R, \\ B + C - 2A &= \beta - \gamma, \\ VW &= vw\cos 2\theta + \tfrac{1}{2}(v^2 - w^2)\sin 2\theta \\ &= R^{-1}\{vw\alpha + u(v^2 - w^2)\}, \\ 4(C-A)(A-B) &= (B-C)^2 - (B+C-2A)^2 \\ &= 4\beta\gamma + 4u^2, \\ 2(V \pm iW)\{V(C-A) &\pm iW(B-A)\} \\ &= (V \pm iW)^2(B+C-2A) - (V^2 + W^2)(B-C) \\ &= e^{\mp 2i\theta}(v \pm iw)^2(\beta - \gamma) - R(v^2 + w^2). \end{aligned}$$

Of the five terms on the right of (12), the fifth gives no trouble since it is evidently equal to $27\{vwa + u(v^2 - w^2)\}^2$. We deal with the first and fourth terms by expressing their sum as the product of conjugate complex factors and observing that

$$\begin{aligned}(B-C)(C-A)(A-B) + V^2(C-A) + W^2(A-B) \pm iVW(B+C-2A)\\ = R(\beta\gamma + u^2) + (V \pm iW)\{V(C-A) \pm iW(B-A)\}\\ = R(\beta\gamma + u^2 - \tfrac{1}{2}v^2 - \tfrac{1}{2}w^2) + \tfrac{1}{2}e^{\pm 2i\theta}(v \pm iw)^2(\beta - \gamma)\\ = e^{\pm 2i\theta}\{(a \mp 2iu)(\beta\gamma + u^2 - \tfrac{1}{2}v^2 - \tfrac{1}{2}w^2) + \tfrac{1}{2}(v \pm iw)^2(\beta - \gamma)\}\\ = e^{\pm 2i\theta}[a\beta\gamma + \Sigma au^2 \mp i\{2u\beta\gamma - vw(\beta - \gamma) + u(2u^2 - v^2 - w^2)\}].\end{aligned}$$

We deal with the sum of the second and third terms on the right of (12) in a similar manner by observing that

$$\begin{aligned}2V\{(B-C)(A-B) + V^2 + W^2\} \pm 2iW\{(B-C)(C-A) + V^2 + W^2\}\\ = (V \pm iW)\{2V^2 + 2W^2 - (B-C)^2\} - (V \mp iW)(B-C)(B+C-2A)\\ = e^{\mp i\theta}(v \pm iw)(2v^2 + 2w^2 - a^2 - 4u^2) - e^{\mp i\theta}(v \mp iw)R(\beta - \gamma)\\ = e^{\mp i\theta}\{(v \pm iw)(2v^2 + 2w^2 - a^2 - 4u^2) - (v \mp iw)(a \mp 2iu)(\beta - \gamma)\}\\ = 2e^{\mp i\theta}[\{va\gamma + uw(\beta - \gamma) + v(v^2 + w^2 - 2u^2)\}\\ \pm i\{wa\beta + uv(\beta - \gamma) + w(v^2 + w^2 - 2u^2)\}].\end{aligned}$$

When we substitute these results in (12), we obtain an identity which is curiously unsymmetrical in appearance; it is comprised in the following theorem.

THEOREM 3. *The general function $F(a, b, c; u, v, w)$ can be represented as a sum of squares by the formula*

$$\begin{aligned}(13)\qquad F(a, b, c; u, v, w) \equiv (a\beta\gamma + \Sigma au^2)^2\\ + 4\{va\gamma + uw(\beta - \gamma) + v(v^2 + w^2 - 2u^2)\}^2\\ + 4\{wa\beta + uv(\beta - \gamma) + w(v^2 + w^2 - 2u^2)\}^2\\ + \{2u\beta\gamma - vw(\beta - \gamma) + u(2u^2 - v^2 - w^2)\}^2\\ + 27\{vwa + u(v^2 - w^2)\}^2.\end{aligned}$$

This identity appears to be new. To deduce from it an identity which is symmetrical in appearance (but of greater length when written in full), change a, b, c and u, v, w cyclically and sum. This process gives

$$\begin{aligned}3F(a, b, c; u, v, w) \equiv 3(a\beta\gamma + \Sigma au^2)^2\\ + 4\Sigma\{u\beta\gamma + vw(a - \beta) + u(u^2 + v^2 - 2w^2)\}^2\\ + 4\Sigma\{u\beta\gamma + vw(\gamma - a) + u(w^2 + u^2 - 2v^2)\}^2\\ + \Sigma\{2u\beta\gamma - vw(\beta - \gamma) + u(2u^2 - v^2 - w^2)\}^2\\ + 27\Sigma\{vwa + u(v^2 - w^2)\}^2.\end{aligned}$$

Now apply the formula

$$4p^2 + 4q^2 = 2(p+q)^2 + 2(p-q)^2$$

to like terms of the sums in the second and third lines of this result. We immediately obtain Kummer's theorem as follows:

THEOREM 4. *The general function* $F(a, b, c; u, v, w)$ *can be represented as a sum of squares by the formula*

$$\begin{aligned}(14)\qquad F(a,b,c;u,v,w) \equiv{}& (a\beta\gamma + \Sigma a u^2)^2 \\ &+ \Sigma\{2u\beta\gamma - vw(\beta-\gamma) + u(2u^2 - v^2 - w^2)\}^2 \\ &+ 15\Sigma\{vwa + u(v^2 - w^2)\}^2.\end{aligned}$$

If we take $a = b = c$ in (14) and compare the result with (5), we obtain an unusual proof of the theorem of the arithmetic and geometric means for three variables.

Since this special case $a = b = c$ is the only special case of (14) which is mentioned by Kummer, it seems highly probable that his method of constructing (14) and the method which I have just given are completely different.

The reader may find it an interesting example to obtain the ordinary conditions for the characteristic equation to have a pair of equal roots directly from (14).

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The Elementary Divisors, Associated with 0, of a Singular M-matrix

By Hans Schneider

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1. Many investigations have been concerned with a square matrix P with non-negative coefficients (elements). It is remarkable that many interesting properties of P are determined by the set Σ of index pairs of positive (*i.e.* non-zero) coefficients of P, the actual values of these coefficients being irrelevant. Thus, for example, the number of characteristic roots equal in absolute value to the largest non-negative characteristic root ρ depends on Σ alone, if P is irreducible. If P is reducible, then Σ determines the standard forms of P (cf. § 3). The multiplicity of ρ depends on Σ, and on the set S of indices of those submatrices in the diagonal in a standard form of P which have ρ as a characteristic root. It has apparently not been considered before whether Σ and S also determine the elementary divisors associated with ρ. We shall show that, in general, the elementary divisors do not depend on these sets alone, but that necessary and sufficient conditions may be found in terms of Σ and S (*a*) for the elementary divisors associated with ρ to be simple, and (*b*) that there is only one elementary divisor associated with ρ.

The square matrix $A = [a_{ij}]$ is called an M-matrix [1] if (1) $a_{ii} \geqq 0$ for all i; (2) $a_{ij} \leqq 0$ when $i \neq j$; and (3) all non-zero characteristic roots of A have positive real part. If $P = [p_{ij}]$ is a square matrix with non-negative coefficients and ρ is its greatest non-negative characteristic root, then $\rho \geqq p_{ii}$, for all i (O. Taussky [7]). Hence $\rho I - P$ is a singular M-matrix. Conversely, if A is a singular M-matrix and $\rho \geqq m_{ii}$ for all i, then $\rho I - A$ is a matrix with non-negative elements. Thus it is equivalent, and rather more convenient, to study the elementary divisors associated with the characteristic root 0 of a singular M-matrix.

2. We shall now explain our notation and terminology, which differ in some respects from the usual ones. We introduce a partial ordering on a set of conformable matrices with real coefficients by setting $A \geqq B$ if $a_{ij} \geqq b_{ij}$ for all i, j, where $A = [a_{ij}]$ and $B = [b_{ij}]$. A second partial ordering is introduced by setting $A \geqq B$, if *either*

[1] The term M-determinant was used by A. Ostrowski [4], [5]. It has been proved [6], p. 19, that our definition is equivalent to Ostrowski's.

$a_{ij} > b_{ij}$ for all i, j, *or* $A = B$. Expressions such as $A (> B$, and $A < B$ then have their natural meanings. If $A > 0$, we call A "strictly positive," if $A (> 0$ we call A "weakly positive" or just "positive." We shall similarly call A negative if $A <) 0$.

The notation used by previous authors (cf. Frobenius [2], Wielandt [8], and others) is less convenient and a little less satisfactory logically. It obscures the fact that we are continually dealing with *two* partial orderings. While $A > B$ has the same meaning in both notations, these authors use $A \geqq B$ in place of $A (\geqq B$. Where we may write "$A (> 0$," they would have to write "$A \geqq 0$ but $A \neq 0$." It is surely unfortunate, also, that in their notation "$A \geqq B$" is not equivalent to "either $A > B$ or $A = B$."

We note that if $A = [a]$ is 1×1, then $A > 0$ is equivalent to $A (> 0$, and if A is identified with a then $A > 0$ has its usual meaning.

Column and row vectors may be regarded as matrices, and the same notation will be employed there.

3. Our principal results will be enumerated in terms of the numbers $R_{ij}(A, P)$ defined below. Let A be a square matrix [1] and let P be the diagonally symmetric partition $[A_{ij}]$, $i, j = 1, \ldots, k$. For $i, j = 1, \ldots, k$ we set

$$r_{ij}(A, P) = 0 \text{ if } i \neq j \text{ and } A_{ij} = 0,$$

and

$$r_{ij}(A, P) = 1 \text{ if } i = j, \text{ or if } A_{ij} \neq 0.$$

Where no confusion can arise we shall write r_{ij} for $r_{ij}(A, P)$. Next we set

$$R_{ij}(A, P) = \max r_{ih} r_{hl} \ldots r_{nj},$$

the maximum being taken over all sequences $(i, h, \ldots, n, j)$. Again we shall generally write R_{ij} for $R_{ij}(A, P)$. For future reference we note that

$$\text{either } R_{ij} = 0 \text{ or } R_{ij} = 1;$$

$$R_{ii} = 1 \text{ for } i = 1, \ldots, k;$$

$$\sum_{h=1}^{k} R_{ih} R_{hj} \geqq R_{ij} \geqq R_{il} R_{lj}, \qquad 1 \leqq l \leqq k; \tag{1}$$

$$\sum_{h=1, h \neq i}^{k} r_{ih} R_{hj} \geqq R_{ij} \geqq \max_{h \neq i} r_{ih} R_{hj} \quad \text{if } i \neq j. \tag{2}$$

[1] The field of the coefficients of A is here immaterial. But in the remaining sections we shall assume that all matrices occurring have real coefficients.

If A is a square matrix, then there exists a permutation matrix, T, for which $A^* = T^{-1}AT$ can be partitioned $[A^*_{ij}]$, $i, j = 1, \ldots, k$ such that (1) $A^*_{ij} = 0$ if $i < j$, (2) A^*_{ii}, $i = 1, \ldots, k$, is irreducible. We shall call A^* a standard form of A, and we shall say that A^* is in standard form. In general, the standard form of A is not unique. If A^0 is a standard form of A, and A^0_{ii}, $i = 1, \ldots, l$, are the irreducible matrices in its diagonal, then $l = k$, and $A^0_{\sigma(i)\sigma(j)} = T_i^{-1} A^*_{ij} T_j$, where σ is a permutation of $(1, \ldots, k)$ and $T_1, \ldots, T_k$ are permutation matrices. Thus there exists a one-one correspondence between the irreducible submatrices in the diagonal of any two standard forms such that corresponding submatrices have the same characteristic roots. In particular, all standard forms have the same number of singular irreducible submatrices in the diagonal.

In view of what is to follow we shall examine the connection between the $R_{ij}(A, P)$ and a standard form of A.

LEMMA 1. *Let P be the partition $[A_{ij}]$, $i, j = 1, \ldots, k$ of the square matrix A such that the A_{ii} in the diagonal are irreducible. Then A is in standard form if and only if $R_{ij} = 0$ whenever $i < j$.*

Proof. We must show that '$R_{ij} = 0$ whenever $i < j$' is equivalent to '$r_{ij} = 0$ whenever $i < j$'. Clearly '$R_{ij} = 0$ whenever $i < j$' implies '$r_{ij} = 0$ whenever $i < j$'. To prove the converse we note that if $i < j$, then any sequence $(i, h, \ldots, n, j)$ contains two consecutive members l, m such that $l < m$. The lemma follows from the definition of R_{ij}.

THEOREM 1. *Let P be the partition $[A_{ij}]$, $i, j = 1, \ldots, k$ of A, where the A_{ii} are irreducible, and let A be in standard form. Let $1 \leqslant \alpha, \beta \leqslant k$. There exists a permutation σ of $(1, \ldots, k)$ for which $A^* = [A_{\sigma(i)\sigma(j)}]$, $i, j = 1, \ldots, k$, is in standard form and $\sigma(\beta) < \sigma(\alpha)$ if and only if $R_{\beta\alpha}(A, P) = 0$.*

Proof. Let P^* be the partition $[A_{\sigma(i)\sigma(j)}]$ of A^* and put $R^*_{ij} = R_{ij}(A^*, P^*)$. We have $R^*_{\sigma(i)\sigma(j)} = R_{ij} = R_{ij}(A, P)$, $i, j = 1, \ldots, k$. Hence by Lemma 1, if A^* is in standard form and $\sigma(\beta) < \sigma(\alpha)$, then $R_{\beta\alpha} = R^*_{\sigma(\beta)\sigma(\alpha)} = 0$.

Conversely let $R_{\beta\alpha} = 0$. Since by (1) $R_{i\alpha} \geqslant R_{\beta i} R_{i\alpha}$ it follows that $R_{\beta i} R_{i\alpha} = 0$ for $i = 1, \ldots, k$. Hence we may partition $(1, \ldots, k)$ into three sets E_1, E_2, E_3 so that $i \epsilon E_1$ if $R_{\beta i} = 1$ and $R_{i\alpha} = 0$; $i \epsilon E_2$ if $R_{\beta i} = R_{i\alpha} = 0$; and $i \epsilon E_3$ if $R_{\beta i} = 0$ and $R_{i\alpha} = 1$. Let σ be the permutation of $(1, \ldots, k)$ for which $\sigma(i) < \sigma(j)$ if $i < j$ and $i \epsilon E_\lambda$, $j \epsilon E_\mu$, with

$\lambda < \mu$; while $\sigma(i) < \sigma(j)$ if $i, j \epsilon E_\lambda$ and $i < j$, where $\lambda = 1, 2, 3$. Let P^* be the partition $[A_{\sigma(i)\sigma(j)}]$, $i, j = 1, \ldots, k$, of A^*. Let $R^*_{ij} = R_{ij}(A^*, P^*)$. Let $i \epsilon E_\lambda$, $j \epsilon E_\mu$, and suppose that $\sigma(i) < \sigma(j)$. Then $\lambda \leqq \mu$. If $\lambda = \mu$, then $i < j$ so that by Lemma 1, $R^*_{\sigma(i)\sigma(j)} = R_{ij} = 0$. If, on the other hand, $\lambda < \mu$, then either $\lambda = 1$ or $\mu = 3$. If $\lambda = 1$, then $R_{\beta i} = 1$, whence $R_{ij} = R_{\beta i} R_{ij} \leqq R_{\beta j} = 0$, since $\mu \geqq 2$. If $\mu = 3$, then $R_{j\alpha} = 1$, whence $R_{ij} = R_{ii} R_{j\alpha} \leqq R_{i\alpha} = 0$, since $\lambda \leqq 2$. We conclude that $R^*_{\sigma(i)\sigma(j)} = R_{ij} = 0$ whenever $\sigma(i) < \sigma(j)$. Thus, by Lemma 1, A^* is in standard form. We need now only prove that $\sigma(\beta) < \sigma(\alpha)$. But $R_{\beta\beta} = 1$, $R_{\beta\alpha} = 0$, $R_{\alpha\alpha} = 1$ imply that $\beta \epsilon E_1$, $\alpha \epsilon E_3$, and the result follows.

4. We now turn to the consideration of M-matrices. If the matrix A is partitioned $[A_{ij}]$, $i, j, = 1, \ldots, k$, we shall assume any column vector x to be conformably partitioned into $(x_1, \ldots, x_k)$.

LEMMA 2. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$ be an M-matrix in standard form. Let $x = (x_1, \ldots, x_k)$ and let*

$$\left.\begin{aligned} x_i &= 0 \text{ when } R_{i\alpha} = 0 \\ x_i &> 0 \text{ when } R_{i\alpha} = 1, \end{aligned}\right\} \tag{3}$$

for $i = 1, \ldots, h-1$, where $h > \alpha$. If

$$\left.\begin{aligned} y_1 &= 0 \\ y_i &= -\sum_{j=1}^{i-1} A_{ij} x_j, \; i = 2, \ldots, k \end{aligned}\right\} \tag{4}$$

then

$$\left.\begin{aligned} y_h &= 0 \quad \text{if } R_{h\alpha} = 0 \\ y_h &> 0 \quad \text{if } R_{h\alpha} = 1 \end{aligned}\right\}. \tag{5}$$

Proof. Clearly $y_h (\geqq 0$; and $y_h = 0$ if and only if $A_{hj} x_j = 0$ for $j = 1, \ldots, h-1$, since $A_{hj} x_j \leqq) 0$ for $j = 1, \ldots, h-1$. Hence, by the assumptions about the x_j, $y_h = 0$ if and only if

$$r_{hj} R_{j\alpha} = 0, \; j = 1, \ldots, h-1. \tag{6}$$

Since $r_{hj} = 0$ when $h < j$ we have $\sum_{j=1}^{h-1} r_{hj} R_{j\alpha} = \sum_{j=1,\, j \neq h}^{k} r_{hj} R_{j\alpha}$ and $\max_{j<h} r_{hj} R_{j\alpha} = \max_{j \neq h} r_{hj} R_{j\alpha}$. Since $h \neq \alpha$, it can now easily be shown from (2) that (6) holds if and only if $R_{h\alpha} = 0$. The lemma follows.

THEOREM 2. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$ be a singular M-matrix in standard form. Let S be the set of indices of singular A_{ii}. If $\alpha \epsilon S$, and*

$R_{\beta\alpha} = 0$ whenever $\beta \in S$, $\beta \neq \alpha$, then there exists a positive characteristic column vector x of A associated with 0 satisfying (3) *for $i = 1, \ldots, k$.*

Proof. Let x be any column vector, and let y satisfy (4). Then $Ax = 0$ if and only if

$$A_{ii}x_i = y_i \tag{7}$$

for $i = 1, \ldots, k$.

Now let $x_i = 0$ when $i < \alpha$. The singular irreducible M-matrix $A_{\alpha\alpha}$ has a strictly positive characteristic vector x_α associated with 0, cf. [2], [4]. As A is in standard form, $R_{i\alpha} = 0$ when $i < \alpha$, $R_{\alpha\alpha} = 1$, and therefore $x_1, \ldots, x_\alpha$ satisfy (3).

Let us suppose inductively that $x_1, \ldots, x_{h-1}$, $h > \alpha$, satisfy (3). If $y_1, \ldots, y_h$ satisfy (4), then y_h also satisfies (5) by Lemma 2. Thus if $R_{h\alpha} = 0$ then $y_h = 0$; and so if $x_h = 0$ then x_h satisfies (7) for $i = h$. If $R_{h\alpha} = 1$, then $y_h (> 0$, and by assumption A_{hh} is non-singular. It is known that the inverse of a non-singular irreducible M-matrix is strictly positive ([2], [4]). Hence if $x_h = A_{hh}^{-1} y_h$, then $x_h > 0$, and x_h satisfies (7). We have thus constructed a vector x_h satisfying (3) and (7), for $i = h$. The theorem follows by induction.

For the sake of completeness we shall prove the well-known Corollary 1.

COROLLARY 1. *A singular M-matrix has a positive characteristic vector associated with* 0.

Proof. Let α be the largest member of S. Then $R_{\beta\alpha} = 0$, whenever $\beta \in S$, $\beta \neq \alpha$, and the corollary follows from Theorem 2.

It is also convenient to state Corollary 2 at this point.

COROLLARY 2. *Let $\gamma_1, \ldots, \gamma_s$ be the members of S. If $R_{\beta\alpha} = 0$ whenever $\alpha, \beta \in S$, $\alpha \neq \beta$, then A has s linearly independent characteristic column vectors $x^1, \ldots, x^s$ associated with* 0, *where x^j satisfies* (3) *with $\alpha = \gamma_j$.*

Proof. Theorem 2 shows the existence of the characteristic vectors x^j, $j = 1, \ldots, s$, satisfying (3) with $\alpha = \gamma_j$. Suppose that $\sum_{h=1}^{s} \lambda_h x^h = 0$. Then, for $i = 1, \ldots, k$, we have $\sum_{h=1}^{s} \lambda_h x_i^h = 0$. Let $\alpha = \gamma_j$. Since $R_{\beta\alpha} = 0$ when $\beta = \gamma_h$, $h \neq j$, it follows that $x_\alpha^h = 0$ if $h \neq j$. Hence $\lambda_j x_\alpha^j = 0$; and $x^j \neq 0$ now implies $\lambda_j = 0$. The linear independence of $x^1, \ldots, x^s$ follows.

5. If $x = (x_1, \ldots, x_k)$ we shall call x_i the ith vector component of x.

LEMMA 3. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let $\gamma_1, \ldots, \gamma_s$, where $\gamma_{j-1} < \gamma_j$, be the members of the set S of indices of singular A_{ii}. If there exist m linearly independent characteristic vectors of A associated with 0, then for each integer n, $n \leqq m$, there are at least n of these vectors such that the ith vector component is non-zero for some $i \leqq \gamma_{n+s-m}$.*

Proof. If $\gamma_{n+s-m} = k$ there is nothing to prove. So let $\gamma_{n+s-m} < k$ and suppose that $x^1, \ldots, x^m$ are linearly independent characteristic vectors associated with 0, such that $x_i^j = 0$, for $i = 1, \ldots, \gamma_{n+s-m}$ and $j = n, \ldots, m$. If $\mu = \gamma_{n+s-m} + 1$, the vectors $(x_\mu^j, \ldots, x_k^j)$, $j = n, \ldots, m$, form $n - m + 1$ linearly independent characteristic vectors associated with 0 of the matrix $B = [A_{ij}]$, $i, j = \mu, \ldots, k$. But the multiplicity of 0 in B equals the number of singular A_{ii} in B, and so equals $m - n$. This yields a contradiction, and the lemma follows.

LEMMA 4. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let $\gamma_1, \ldots, \gamma_s$, where $\gamma_{j-1} < \gamma_j$, be the members of S. If A has s linearly independent characteristic column vectors associated with 0, then there exists a set $x^1, \ldots, x^s$ of such vectors for which*

$$\left.\begin{aligned} x_i^j &= 0 \quad \text{if } i < \gamma_j \\ \text{but} \qquad x_i^j &\neq 0 \quad \text{if } i = \gamma_j \end{aligned}\right\} \tag{8}$$

for $j = 1, \ldots, s$.

Proof. Let $z^1, \ldots, z^s$ be linearly independent characteristic vectors associated with 0. If $z_i^j = 0$ if $i < \delta_j$ but $z_i^j \neq 0$ if $i = \delta_j$, then $A_{ii} z_i^j = 0$ for $i = \delta_j$. Hence $\delta_j \epsilon S$. Thus $z_i^j = 0$ if $i < \gamma_1$, for $j = 1, \ldots, s$. It also follows from Lemma 3, with $m = s$, $n = 1$, that for some j we have $z_i^j \neq 0$, if $i = \gamma_1$. We may therefore assume inductively that we have linearly independent characteristic vectors $x^1, \ldots, x^n, z^{n+1}, \ldots, z^s$, associated with 0, such that (*a*) (8) holds for $j = 1, \ldots, n$ and (*b*) $z_i^j = 0$ if $i < \gamma_n$, for $j = n+1, \ldots, s$. Let $\alpha = \gamma_n$. Then $A_{\alpha\alpha} x_\alpha^n = A_{\alpha\alpha} z_\alpha^j = 0$, $j = n+1, \ldots, s$. Since an irreducible singular

M-matrix has only one linearly independent characteristic column vector associated with 0, it follows that $z_\alpha^j = \lambda_j x_\alpha^n$, $j = n+1, \ldots, s$. Let $x^j = z^j - \lambda_j x^n, j = n+1, \ldots, s$. The vectors $x^1, \ldots, x^s$ are linearly independent characteristic vectors associated with 0, and, for $j = n+1, \ldots, s$, $x_i^j = 0$ if $i \leqq \gamma_n$. Hence by the remark at the beginning of the proof, $x_i^j = 0$ if $i < \gamma_{n+1}$ for $j = n+1, \ldots, s$. It follows from Lemma 3 that there is a $j \geqq n+1$ such that $x_i^j \neq 0$, if $i = \gamma_{n+1}$. Suppose this $j = n+1$. Then (*a*) (8) holds for $j = 1, \ldots, n+1$, and (*b*) $x_i^j = 0$ if $i < \gamma_{n+1}$, for $j = n+2, \ldots, s$. The lemma follows by induction.

6. The following lemma is of some interest in itself. It is related to a theorem of Collatz [1], and other results on positive irreducible matrices.

LEMMA 5. *Let A be an irreducible singular M-matrix and let Ax ($\geqq 0$ or 0) $\geqq Ax$. Then $Ax = 0$.*

Proof. Let Ax ($\geqq 0$ or $Ax \leqq$) 0, and let u' be the strictly positive characteristic row vector of A, associated with 0. If either z (> 0 or $z <$) 0, then either $u'z > 0$ or $u'z < 0$. Hence $Ax = 0$.

We now come to one of our main theorems.

THEOREM 3. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let S be the set of indices of singular A_{ii}. The elementary divisors associated with the characteristic root 0 are all linear if and only if $R_{\beta\alpha} = 0$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$.*

Proof. Let S have the s members $\gamma_1, \ldots, \gamma_s$ where $\gamma_{j-1} < \gamma_j$. The elementary divisors associated with 0 are all linear if and only if that characteristic root has s linearly independent characteristic vectors associated with it.

If $R_{\beta\alpha} = 0$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$ then by Corollary 2 to Theorem 2, 0 has s linearly independent characteristic vectors associated with it. Suppose, conversely, that 0 has the s linearly independent characteristic vectors $x^1, \ldots, x^s$ associated with it. By Lemma 3, we may assume that $x^1, \ldots, x^s$ satisfy (S). Let us assume that for some $\alpha, \beta \in S$, $\alpha \neq \beta$, we have $R_{\beta\alpha} = 1$. We shall obtain a contradiction. We may choose α, β so that $R_{\beta\alpha} = 1, \beta > \alpha$ and $\beta - \alpha \leqq \beta' - \alpha'$ for all $\alpha', \beta' \in S$, $\alpha' \neq \beta'$, for which $R_{\beta'\alpha'} = 1$. If $\alpha \leqslant \gamma \leqq \delta \leqq \beta$ and $\gamma, \delta \in S$ then $R_{\delta\gamma} = 0$ unless $\gamma = \alpha$ and $\delta = \beta$. Let

B be the matrix $[A_{ij}]$, $i, j = \alpha, \ldots, \beta - 1$. Let $\delta_1, \delta_2, \ldots, \delta_r$ be the indices of the singular A_{ii} of B in ascending order of magnitude. Thus $\delta_1 = \alpha = \gamma_j$, say. We deduce from Corollary 2 to Theorem 2 that B has r linearly independent characteristic vectors $(z^h_\alpha, \ldots, z^h_{\beta-1})$, $h = 1, \ldots, r$, associated with 0, where z^h_i satisfies (3) (provided we replace α by δ_h there), for $i = \alpha, \ldots, \beta - 1$. Since the multiplicity of 0 in B is r, any characteristic vector of B associated with 0 is a linear combination of these. Since $\alpha = \gamma_j$, $(x^j_\alpha, \ldots, x^j_{\beta-1})$ is a characteristic vector of B associated with 0. Hence $x^j_i = \sum_{h=1}^{r} \lambda_h z^h_i$, for $i = \alpha, \ldots, \beta - 1$. Further, $\lambda_1 \neq 0$ since $x^j_\alpha \neq 0$ but $z^h_\alpha = 0$ if $h = 2, \ldots, r$. It follows that $A_{\beta\beta} x^j_\beta = \sum_{h=1}^{r} \lambda_h y^h_\beta$ where $y^h_\beta = -\sum_{i=\alpha}^{\beta-1} A_{\beta i} z^h_i$, $h = 1, \ldots, r$. On putting $z^h_i = 0$ when $i = 1, \ldots, \alpha - 1$, we obtain from Lemma 2 that $y^h_\beta (> 0$ if $R_{\beta\gamma} = 1$, but $y^h_\beta = 0$ if $R_{\beta\gamma} = 0$, where $\gamma = \delta_h$. Hence $y^1_\beta (> 0$, but $y^h_\beta = 0$, for $h = 2, \ldots, r$. Thus $A_{\beta\beta} x^j_\beta = \lambda_1 y'_\beta$, and so either $A_{\beta\beta} x^j_\beta (> 0$ or $0 (> A_{\beta\beta} x^j_\beta$. But this is not possible, by Lemma 5. It follows that $R_{\beta\alpha} = 0$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$. The theorem is proved.

In view of Theorem 1 we obtain immediately

COROLLARY 1. *The elementary divisors associated with* 0 *are all linear if and only if for each* $\alpha \in S$ *there exists a permutation* σ *of* $(1, \ldots, k)$ *such that* $[A_{\sigma(i)\sigma(j)}]$, $i, j = 1, \ldots, k$, *is in standard form, and* $\sigma(\beta) \leqq \sigma(\alpha)$ *for any* $\beta \in S$.

The square matrix $A = [a_{ij}]$ is called *Minkowskian* if (1) $a_{ii} \geqq 0$ for all i, (2) $a_{ij} \leqq 0$, when $i \neq j$, and (3) $\sum_j a_{ij} \geqq 0$ for all i. A Minkowskian matrix is an M-matrix: cf. [4], [6].

COROLLARY 2. *Let* A *be a singular Minkowskian matrix. The elementary divisors associated with* 0 *are all linear.*

Proof. We may assume that $A = [A_{ij}]$, $i, j = 1, \ldots, k$, is in standard form. Let S be as above. If $C = [c_{ij}]$ is an irreducible Minkowskian matrix, then C is singular if and only if $\sum_j c_{ij} = 0$ for all i, [4]. Hence if $\alpha \in S$, then $A_{\alpha j} = 0$, provided that $j \neq \alpha$. It follows that $R_{\alpha j} = 0$, if $j \neq \alpha$. The corollary now follows from Theorem 2.

In the case when $\sum_j a_{ij} = 0$ for all i, this result has already been proved by Ledermann [3].

7. Results similar to those we have found for characteristic column vectors may be stated for characteristic row vectors. Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be in standard form. The transposed matrix $A' = [A'_{ij}]$, $i, j, = 1, \ldots, k$ (where $A'_{ij} = (A_{ji})'$) is not necessarily in standard form. However, if σ is the permutation for which $\sigma(i) = k + 1 - i$, $i = 1, \ldots, k$, then $B = [A'_{\sigma(i)\sigma(j)}]$, $i, j = 1, \ldots, k$, is in standard form. Let P, P', and Q be the partition described above of A, A' and B respectively. Then $R_{\sigma(j)\sigma(i)}(B, Q) = R_{ji}(A', P') = R_{ij}(A, P)$. To any characteristic row vector $(u'_1, \ldots, u'_k)$ of A, associated with 0, there corresponds the characteristic column vector $(u_{\sigma(1)}, \ldots, u_{\sigma(k)})$ of B associated with 0. We may deduce Theorem 2a and Corollary 2a, from Theorem 2 and Corollary 2.

THEOREM 2a. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let S be the set of indices of singular A_{ii}. If $\alpha \epsilon S$ and $R_{\alpha\beta} = 0$ whenever $\beta \epsilon S$ and $\beta \neq \alpha$, then there exists a positive characteristic row vector $u' = (u_1', \ldots, u'_k)$ associated with 0, satisfying*

$$\left.\begin{aligned} u_i &= 0 \quad \text{when } R_{\alpha i} = 0 \\ u_i &> 0 \quad \text{when } R_{\alpha i} = 1 \end{aligned}\right\}, \tag{9}$$

for $i = 1, \ldots, k$.

COROLLARY 2a. Let $\gamma_1, \ldots, \gamma_s$ *be the members of S. If $R_{\alpha\beta} = 0$ whenever $\alpha, \beta \epsilon S$ and $\alpha \neq \beta$ then A has s linearly independent characteristic row vectors $u'^1, \ldots, u'^s$ associated with 0, where u'^j satisfies (9) with $\alpha = \gamma_j$.*

THEOREM 4. *Let A be a singular M-matrix. If the elementary divisors associated with the characteristic root 0 are all linear, then the principal idempotent element associated with 0 is positive.*

Proof. Let C be a matrix whose characteristic root ω, of multiplicity s, has only linear elementary divisors associated with it. There exist linearly independent characteristic column vectors $x^1, \ldots, x^s$ and linearly independent characteristic row vectors $u'^1, \ldots, u'^s$ associated with ω, such that $u'^h x^j = \delta_{hj}$, $h, j = 1, \ldots, s$, the Kronecker delta. The principal idempotent element associated with ω is the matrix $\sum_{h=1}^{s} x^h u'^h$.

Let P^* be the partition $A^* = [A^*_{ij}]$, $i, j = 1, \ldots, k$, of a standard form of the singular M-matrix A with only linear elementary divisors associated with 0, and let $S = (\gamma_1, \ldots, \gamma_s)$ be the set of indices of singular A^*_{ii}. By Theorem 3, $R^*_{\beta\alpha} = R_{\beta\alpha}(A^*, P^*) = 0$, when $\alpha, \beta \epsilon S$, $\alpha \neq \beta$. Hence by Corollary 2 to Theorem 2 there exist linearly independent characteristic column vectors $x^1, \ldots, x^s$ associated with 0 such that x^j satisfies (3) with $\alpha = \gamma_j$. Similarly, by Corollary 2a to Theorem 2a there exist linearly independent characteristic row vectors $u'^1, \ldots, u'^s$, associated with 0, such that u'^j satisfies (9) with $\alpha = \gamma_j$. Let $\alpha = \gamma_j$ and $\beta = \gamma_h$. Since $u'^h x^j = \sum_{i=1}^{k} u'^h_i x^j_i$ and $u'^h_i x^j_i = 0$ if and only if $R^*_{\beta i} R^*_{i\alpha} = 0$ it follows that $u'^h x^j \geqq 0$, and that $u'^h x^j = 0$ if and only if $\sum_{i=1}^{k} R^*_{\beta i} R^*_{i\alpha} = 0$. But we may deduce from (1) that $\sum_{i=1}^{k} R^*_{\beta i} R^*_{i\alpha} = 0$ if and only if $R^*_{\beta\alpha} = 0$. Hence $u'^h x^j = 0$ if $h \neq j$, but $u'^j x^j > 0$, for $h, j = 1, \ldots, s$. We may clearly assume that $u'^1, \ldots, u'^s$ have been multipled by positive factors so that $u'^h x^j = \delta_{hj}$. Then the idempotent element of A^* associated with 0 is $E^* = \sum_{j=1}^{s} x^j u'^j$. Thus $E^* (\geqq 0$. If E is partitioned conformably with A, then $E_{ii} = \sum_{h=1}^{s} x^h_i u'^h_i$, $i = 1, \ldots, k$. Since $u'^j_a > 0$, and $x^j_a > 0$, but $u'^h_a = 0$, and $x^h_a = 0$, when $h \neq j$, it follows that $E^*_{aa} = x^h_a u'^h_a > 0$. Hence $E^* (> 0$.

The principal idempotent element E of A associated with 0 is obtained from that of A^* by means of a transformation by a permutation matrix. Hence E, too, is positive.

8. We have already remarked that the elementary divisors of a matrix A, associated with the characteristic root 0 of multiplicity s, are all linear if and only if there are s linearly independent characteristic vectors associated with 0. In the next two sections we shall discuss the other extreme case when there is only one elementary divisor associated with the characteristic root 0 of a singular M-matrix. Equivalent conditions are (a) that 0 has only one linearly independent characteristic vector associated with it; *or* (b) that there exists a set $x^1, \ldots, x^s$ of column vectors such that

$$Ax^j = x^{j+1}, j = 1, \ldots, s-1; \; Ax^s = 0, \text{ and } x^s \neq 0, \tag{10}$$

where s is again the multiplicity of 0.

LEMMA 6. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let $S = (\gamma_1, \ldots, \gamma_s)$, where $\gamma_{j-1} < \gamma_j$, be the set of indices of singular A_{ii}. If $x^1, \ldots, x^s$ is a set of column vectors satisfying* (10), *then x^j, $j = 1, \ldots, s$, satisfies* (8).

Proof. For $j = 1, \ldots, s$, let $x_i^j = 0$ if $i < \delta_j$, but $x_i^j \neq 0$ if $i = \delta_j$. Since $Ax^s = 0$, it follows, as in the proof of Lemma 4, that $\delta_s \epsilon S$. Let us assume inductively that (*a*) $\delta_j \epsilon S$ for $j = h, \ldots, s$; and that (*b*) $\delta_{j-1} < \delta_j$ for $j = h+1, \ldots, s$. These assumptions hold for $h = s$. If y_i^j satisfies (4), when $x_i = x_i^j$, then $Ax^j = x^{j+1}$ if and only if

$$A_{ii}x_i^j = y_i^j + x_i^{j+1}, \tag{11}$$

for $i = 1, \ldots, k$. Let $\beta = \delta_h$. Then $y_\beta^h = 0$; and we may deduce from (*b*) that $x_\beta^{h+1} = 0$. Hence (11) holds for $i = \beta$ and $j = h$ if and only if $A_{\beta\beta}x_\beta^h = 0$. Hence either $x_\beta^h > 0$ or $x_\beta^h < 0$. But $A_{\beta\beta}x_\beta^{h-1} = y_\beta^{h-1} + x_\beta^h$. It follows by Lemma 5 that (*a*) and (*b*) imply that $y_\beta^{h-1} \neq 0$. Thus $\delta_{h-1} < \beta = \delta_h$. We deduce that if $\alpha = \delta_{h-1}$ then $x_\alpha^h = y_\alpha^{h-1} = 0$. Hence by (11) $A_{\alpha\alpha}x_\alpha^{h-1} = 0$, and so $\delta_{h-1} \epsilon S$. By induction we obtain that $\delta_j \epsilon S$, $j = 1, \ldots, s$, and that $\delta_{j-1} < \delta_j$, $j = 2, \ldots, s$. Hence $\delta_j = \gamma_j$, $j = 1, \ldots, s$ and the lemma is proved.

9. LEMMA 7. *Let A be an irreducible M-matrix. Let z and y, where y (> 0, be column vectors conformable with A. Then there exist a real λ and a column vector x such that $Ax = \lambda y + z$.*

Proof. If A is non-singular, then there exists such an x for any λ. Suppose A singular. Since 0 is a simple characteristic root of A, the nullity of A is 1. By Lemma 5, y is linearly independent of the columns of A. Hence any column vector is a linear combination of the columns of A and y. The lemma follows.

THEOREM 5. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let S be the set of indices of singular A_{ii}. There is only one elementary divisor associated with the characteristic root 0 of A if and only if $R_{\beta\alpha} = 1$, whenever $\alpha, \beta \epsilon S$ and $\beta > \alpha$.*

Proof. Suppose that there is only one elementary divisor associated with 0. Let $x^1, \ldots, x^s$ be a set of column vectors satisfying (10). By Lemma 6, x^j satisfies (8), $j = 1, \ldots, s$. Let $\alpha = \gamma_{h-1}$, $\beta = \gamma_h$. The

conditions (a) and (b) of the proof of Lemma 6 are clearly satisfied, and so $y_\beta^{h-1} = -\sum_{i=1}^{\beta-1} A_{ii}x_i^{h-1} = -\sum_{i=\alpha}^{\beta-1} A_{ii}x_i^{h-1} \neq 0$. But $(x_\alpha^{h-1}, \ldots, x_{\beta-1}^{h-1})$ is the one linearly independent characteristic column vector associated with 0 of $B=[A_{ij}]$, $i, j=\alpha, \ldots, \beta-1$, since $x_i^h = 0$, for $i=\alpha, \ldots, \beta-1$. It therefore follows from Theorem 2 that x_i^h, $i=\alpha, \ldots, \beta-1$, satisfies (3), provided that $x^1, \ldots, x^s$ have been multiplied by -1 if necessary. Therefore x_i^h, $i=1, \ldots, \beta-1$, satisfy (3). Hence, by Lemma 2, it follows from $y_\beta^{h-1} \neq 0$ that $R_{\beta\alpha}=1$, when $\alpha=\gamma_{h-1}$, $\beta=\gamma_h$. This is a particular case of the required result. To deduce the general case, let $\alpha=\gamma_j$, $\beta=\gamma_h$, where $h>j$. Then $R_{\beta\alpha} \geqslant R_{\gamma_h\gamma_{h-1}} \ldots R_{\gamma_{j+1}\gamma_j}=1$. Hence $R_{\beta\alpha}=1$, whenever $\alpha, \beta \epsilon S$ and $\beta>\alpha$.

Conversely, let us suppose that $R_{\beta\alpha}=1$ whenever $\alpha, \beta \epsilon S$ and $\beta>\alpha$. By Theorem 2, Corollary 1, there exists a characteristic column vector x^s, associated with 0, which satisfies (8). Let us suppose that there exist column vectors x^j satisfying (8), $j=h, \ldots, s$, such that

$$Ax^j = x^{j+1},\ j=h, \ldots, s-1,\ Ax^s=0,\ x^s \neq 0. \tag{12}$$

We shall construct a vector x^{h-1} satisfying (8) and $Ax^{h-1}=x^h$. Let $\alpha=\gamma_{h-1}$, $\beta=\gamma_h$. Let $x_i^{h-1}=0$, $i=1, \ldots, \alpha-1$, and let $(x_\alpha^{h-1}, \ldots, x_{\beta-1}^{h-1})$ be the one linearly independent characteristic vector, associated with 0, of $B=[A_{ij}]$, $i, j=\alpha, \ldots, \beta-1$. Then x_i^{h-1}, $i=1, \ldots, \beta-1$, satisfies (3) provided that the x_i^{h-1} have all been multiplied by -1, if necessary. If, for $i=1, \ldots, \beta$, y_i^{h-1} is chosen to satisfy (4) with x replaced there by x^{h-1}, then it follows by Lemma 2 that $y_\beta^{h-1}(>0$. Hence by Lemma 7, there exist a λ and an x_β^{h-1} such that $A_{\beta\beta}x_\beta^{h-1} = \lambda y_\beta^{h-1} + x_\beta^h$. We now write x_i^{h-1} for λx_i^{h-1}, $i=1, \ldots, \beta-1$, λy_i^{h-1} for y_i^{h-1}, $i=1, \ldots, \beta$, and leave x_β^{h-1} unchanged. Since $x_i^h=0$, when $i<\beta$, x_i^{h-1}, $i=1, \ldots, \beta$ satisfies (11) with $j=h-1$.

Let us assume inductively that x_i^{h-1}, $i=1, \ldots, l-1$, satisfies (11), where $l>\beta$. We must consider two cases: $l \notin S$ and $l \epsilon S$. If $l \notin S$, then (11) is satisfied when $x_l^{h-1} = A_{ll}^{-1}(y_l^{h-1} + x_l^h)$. If $l \epsilon S$, say $l=\gamma_{h+m}$ (where clearly $m>0$) then there exist, by Lemma 7, a λ and an x_l^{h-1} for which $A_{ll}x_l^{h-1} = y_l^{h-1} + x_l^h + \lambda x_l^{h+m}$, since either $x_l^{h+m}>0$ or $x_l^{h+m}<0$. We now replace x^j by $x^j + \lambda x^{j+m}$, $j=h, \ldots, s$, where, by convention, $x^{j+m}=0$ if $j+m>s$. Then $x^h, \ldots, x^s$ again satisfy (12) and since the original $x_i^{h+m}=0$ when $i<l$, it follows that $x_1^{h-1}, \ldots, x_l^{h-1}$

satisfy (11) with $j = h-1$. By induction we obtain a vector $x^{h-1} = (x_1^{h-1}, \ldots, x_k^{h-1})$ satisfying (11) with $j = h-1$. Thus in addition to (12) we have $Ax^{h-1} = x^h$, where x^{h-1} satisfies (8). Using induction again we obtain a set of vectors satisfying (10) and the theorem is proved.

In virtue of Theorem 1 we obtain the following corollary.

COROLLARY. *Let γ be the largest member of S. There is only one elementary divisor associated with the characteristic root 0 of A if and only if for any permutation σ of $(1, \ldots, k)$ for which $[A_{\sigma(i)\sigma(j)}]$, $i, j = 1, \ldots, k$, is in standard form, we have $\sigma(\gamma) \geqslant \sigma(\alpha)$ whenever $\alpha \epsilon S$.*

An argument along the lines of the second half of the proof of Theorem 5 would lead to the following result, which we shall enunciate as a theorem, though we shall omit the proof. First we should have to generalise Lemma 2.

THEOREM 6. *Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let $S = (\gamma_1, \ldots, \gamma_s)$, $\gamma_{j-1} < \gamma_j$, be the set of indices of singular A_{ii}. If there is only one elementary divisor associated with 0 then there exist positive column vectors $z^1, \ldots, z^s$ such that $x^1, \ldots, x^s$ satisfy (10) if $x^j = (-1)^j z^j$, $j = 1, \ldots, s$. In fact*

$$z_i^h = 0 \text{ when } \sum_{j=h}^{s} R_{i\gamma_j} = 0,$$

and

$$z_i^h > 0 \text{ when } \sum_{j=h}^{s} R_{i\gamma_j} > 0.$$

The standard forms of A depend only on the set Σ of index pairs of non-zero coefficients of A. It is clearly decided by Σ and S whether (*a*) $R_{\beta\alpha} = 0$, whenever $\alpha, \beta \epsilon S$, and $\alpha \neq \beta$; or (*b*) $R_{\beta\alpha} = 1$, whenever $\alpha, \beta \epsilon S$ and $\beta > \alpha$. Let P be a positive matrix, with largest characteristic root ρ. The degrees of the elementary divisors of P associated with ρ are the same as those of the elementary divisors, associated with 0, of the M-matrix $\rho I - P$. Hence Theorems 3 and 5 fulfil the claims of § 1.

10. Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let $S = (\gamma_1, \ldots, \gamma_s)$, $\gamma_{j-1} < \gamma_j$, be as usual, the set of indices of singular A_{ii}. If $s = 1$, then there is clearly only one elementary divisor, associated with 0, and it is linear. If $s = 2$, then there are two elementary divisors of degree 1, or one of degree 2, associated with 0, according as $R_{\beta\alpha} = 0$ or $R_{\beta\alpha} = 1$, where $\alpha = \gamma_1$, and $\beta = \gamma_2$. Suppose that $s = 3$ and let $\gamma_1 = \alpha$, $\gamma_2 = \beta$ and $\gamma_3 = \gamma$. Since

$R_{\alpha\beta} = R_{\alpha\gamma} = R_{\beta\gamma} = 0$, and since $R_{\gamma\beta} = R_{\beta\alpha} = 1$ implies that $R_{\gamma\alpha} = 1$, we may consider seven cases according as $R_{\gamma\beta}$, $R_{\gamma\alpha}$, $R_{\beta\alpha}$ are 1 or 0. The cases $R_{\gamma\beta} = R_{\gamma\alpha} = R_{\beta\alpha} = 0$, and $R_{\gamma\beta} = R_{\gamma\alpha} = R_{\beta\alpha} = 1$, are covered by Theorems 3 and 5. We deduce that in the five other cases 0 must have associated with it one elementary divisor of degree 1 and one of degree 2. This means that A has two linearly independent column vectors z^1, z^2 associated with 0, and that there exists a vector x satisfying either $Ax = z^1$ or $Ax = z^2$. By considering each of the five cases separately it is possible to demonstrate the existence of these vectors without any appeal to Theorems 3 and 5.

As an example we shall consider the case $R_{\gamma\beta} = R_{\gamma\alpha} = 1$, $R_{\beta\alpha} = 0$. Though we shall use the theorems proved previously it would be possible to use special cases of these results which could be proved more simply. It follows from Theorem 5, applied to $[A_{ij}]$, $i, j = \beta, \ldots, k$, that A has a characteristic vector x^3, associated with 0, for which $x^3_i = 0$ when $i < \gamma$, and that there exists an x^2 satisfying $Ax^2 = x^3$. Let $(z^j_1, \ldots, z^j_{\gamma-1})$, $j = 1, 2$, be the two linearly independent characteristic column vectors, associated with 0, of $[A_{ij}]$, $i, j = 1, \ldots, \gamma - 1$. The existence of these vectors is shown by Theorem 3. By Theorem 1, Corollary 2, we may assume that $z^j_i = 0$ if $R_{i\gamma_j} = 0$ and $z^j_i > 0$ if $R_{i\gamma_j} = 1$, for $j = 1, 2$. Hence, by Lemma 2, $w^j_\gamma (> 0, j = 1, 2$, where $w^j_\gamma = -\sum_{h=1}^{\gamma-1} A_{\gamma h} z^j_h$. By Lemma 7, there exist a λ and an x^1_γ satisfying

$$A_{\gamma\gamma}\, x^1_\gamma = w^1_\gamma + \lambda\, w^2_\gamma . \tag{13}$$

Let $x^1_i = z^1_i + \lambda z^2_i$, $i = 1, \ldots, \gamma - 1$. Then $(x^1_1, \ldots, x^1_\gamma)$ is non-zero and, for $i = 1, \ldots, \gamma$, x^1_i satisfies (7) provided y^1_i is chosen to satisfy (4). Since A_{ii} is non-singular if $i > \gamma$, it is easy to establish the existence of a vector $x^1 = (x^1_1, \ldots, x^1_k)$ where x^1_i satisfies (7) for $i = 1, \ldots, k$. Thus x^1 and x^3 are characteristic column vectors associated with 0, which are linearly independent as $x^1_\alpha = z^1_\alpha + \lambda z^2_\alpha = z^1_\alpha > 0$, while $x^3_\alpha = 0$; and $Ax^2 = x^3$.

It follows from Lemma 5 that $\lambda < 0$ in (13). Hence $x^1_\beta = z^1_\beta + \lambda z^2_\beta = \lambda z^2_\beta < 0$, but $x^1_\alpha > 0$, as already noted. Hence x^1 is neither positive nor negative. It is easily established that this property is shared by one of any two linearly independent characteristic vectors, associated

with 0, of the matrix A we have been considering. It is due to this that S and Σ do not necessarily completely determine the elementary divisors associated with 0 of a singular M-matrix when $s > 3$. Thus, in both B and C below, $k = 4$, $S = (1, 2, 3, 4)$, $s = 4$ and $R_{43} = R_{21} = 0$, $R_{42} = R_{41} = R_{32} = R_{31} = 1$.

$$B = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdot & \cdot \\ -1 & -1 & \cdot & \cdot \end{bmatrix} \qquad C = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \end{bmatrix}.$$

But B is of rank 1, C of rank 2. Hence the elementary divisors associated with 0 differ for the two matrices. We also note that the principal idempotent element, associated with 0, of both B and C is the unit matrix, which is, of course, positive. It follows that the converse of Theorem 4 does not hold.

Most of the results of this paper are contained in my 1952 Ph.D. thesis, which was written under the supervision of Professor A. C. Aitken. My thanks are due to Professor Aitken for his great encouragement.

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Note on the Problem of the Electrified Disc

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1. Prof. E. T. Copson [1] has discussed the well-known problem of a circular disc kept at a constant potential V_0 in an external field of potential Φ by reducing it to the solution of two integral equations. The solution is however fairly simple if we use oblate spheroidal co-ordinates. This is due to the fact that in this system of co-ordinates the disc can be represented in terms of one co-ordinate only. This method is applied to the above problem and Copson's results are obtained. The solution when V_0 is not constant, but any surface function of the disc, is also obtained.

2. The relations between cylindrical polar and oblate spheroidal co-ordinates ξ, ζ are

$$z = a\xi\zeta, \ \rho^2 = a^2(1-\xi^2)(1+\zeta^2), \qquad -1 \leqq \xi \leqq 1, 0 \leqq \zeta < \infty .$$

The circular disc $z = 0, \rho \leqq a$ is given by $\zeta = 0$. At the surface of the disc it can easily be seen that

$$\rho^2 = a^2(1-\xi^2), \qquad \frac{\partial V}{\partial z} = \frac{1}{a\xi}\frac{\partial V}{\partial \zeta}.$$

Solutions of Laplace's equation in this system of co-ordinates are oblate spheroidal harmonics and are of the form

$$[A_n^m P_n^m(\xi) + B_n^m Q_n^m(\xi)]\,[A'^m_n P_n^m(i\zeta) + B'^m_n Q_n^m(i\zeta)]\, e^{im\phi},$$

$P_n^m(u)$ and $Q_n^m(u)$ being Legendre associated functions of the first and second kinds respectively.

The potential of any external field which is bounded in the neighbourhood of the disc and at its axis $\xi = 1$ will contain harmonics of the form

$$E_n^m P_n^m(\xi) P_n^m(i\zeta) \cos m\phi,$$

$n + m$ being necessarily even, since otherwise the potential would vanish at the surface of the disc.

By similar considerations V, the potential of the induced charge, will contain only harmonics of the form

$$I_n^m P_n^m(\xi) Q_n^m(i\zeta) \cos m\phi, \qquad n + m \text{ even}.$$

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[1] "On the problem of the electrified disc," *Proc. Edin. Math. Soc.* (2), 8 (1947), 14-19.

If for simplicity we take Φ as one harmonic only, then, since V has the value V_0 at $\zeta = 0$, its value at any point will be

$$V = V_0\left(P_0^0(\xi)\,Q_0^0(i\zeta)/Q_0^0(i0)\right) - E_n^m P_n^m(\xi)\,Q_n^m(i\zeta) \times\left(P_n^m(i0)/Q_n^m(i0)\right)\cos m\phi$$

$$= \frac{2}{\pi}\left\{V_0 \cot^{-1}\xi - iE_n^m P_n^m(\xi)\,Q_n^m(i\zeta)\cos m\phi\right\}.$$

The density of the induced charge, σ, is given by

$$\sigma = -\frac{1}{2\pi}\left[\frac{\partial V}{\partial z}\right]_{\zeta=0} = -\frac{1}{2\pi a\xi}\left[\frac{\partial V}{\partial \zeta}\right]_{\zeta=0}$$

$$= -\frac{1}{\pi^2 a\xi}\left[\frac{-V_0}{1+\zeta^2} + E_n^m P_n^m(\xi)\,Q_n^{m\prime}(i\zeta)\cos m\phi\right]_{\zeta=0}$$

$$= \frac{1}{\pi^2 a\xi}\left\{V_0 - (-1)^{\frac{1}{2}(n-m)}\,\frac{2.4.6\ldots(n+m)}{1.3.5\ldots(n-m-1)}\,E_n^m P_n^m(\xi)\cos m\phi\right\}.$$

The first two special cases considered by Copson follow immediately.

(i) If there is no external field, $E_n^m = 0$ and

$$\sigma = \frac{V_0}{\pi^2\sqrt{(a^2-\rho^2)}}.$$

(ii) If $V_0 = 0$, $\Phi = \rho\cos\phi = -ai\,P_1^1(\xi)\,P_1^1(i\zeta)\cos\phi$, then

$$\sigma = \frac{-ai}{\pi^2 a\xi}\,P_1^1(\xi)\,Q_1^{1\prime}(i0)\cos\phi$$

$$= \frac{-i}{\pi^2\xi}\sqrt{(1-\xi^2)}\,2i\cos\phi = \frac{2}{\pi^2}\frac{\rho\cos\phi}{\sqrt{(a^2-\rho^2)}}.$$

Probably this is not the best method for the last application.

It will be seen that the same method is applicable when the potential at the surface of the disc (which will be taken now as non-conducting) is any surface function. Any such function can be expanded in the form

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m P_n^m(\xi)\,e^{im\phi}.$$

The potential at any external point (ξ, ζ, ϕ) is then given by

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m P_n^m(\xi)\left(Q_n^m(i\zeta)/Q_n^m(i0)\right)e^{im\phi}.$$

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On Integral Transforms

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1. Titchmarsh and others [4] have studied integral transform pairs of the type

$$f(x) = \int_0^\infty k(xy)\, g(y)\, dy,$$

$$g(y) = \int_0^\infty l(xy)\, f(x)\, dx,$$

showing how such pairs may be generated as a consequence of the relation which holds between the Mellin transforms of the kernels.

There is an interesting allied formal process which, in some respects, may be regarded as a generalisation of this.

For a particular transformation T let the basic formulae be

$$H_T(x) = \int_{a_T}^{b_T} k_T(x, y)\, h(y)\, dy, \tag{1}$$

$$h(y) = \int_{c_T}^{d_T} l_T(x, y)\, H_T(x)\, dx. \tag{2}$$

Consider the function

$$f(s) = \int_\alpha^\beta g(s, t)\, i(t)\, dt. \tag{3}$$

Then, formally, we have

$$F_T(u) = \int_\alpha^\beta G_T(u, t)\, i(t)\, dt. \tag{4}$$

Suppose, however, that $G_T(u, t) \equiv \lambda(u)\ k_Q\{\mu(u), t\}$ and also that the path of integration in (4) is identical with, or capable of deformation without crossing a pole of the integrand into, the path from a_Q to b_Q required by the formula (1) for the transformation Q. Then (4) is the same as

$$\frac{F_T(u)}{\lambda(u)} = \int_{a_Q}^{b_Q} k_Q\{\mu(u), t\}\, i(t)\, dt = I_Q\{\mu(u)\}. \tag{5}$$

It follows from the substitution $v = \mu(u)$ (or $u = \mu^{-1}(v)$) that

$$I_Q(v) = \frac{F_T\{\mu^{-1}(v)\}}{\lambda\{\mu^{-1}(v)\}}, \tag{6}$$

and hence

$$i(t) = \int_{c_Q}^{d_Q} l_Q(v, t) \frac{F_T\{\mu^{-1}(v)\}}{\lambda\{\mu^{-1}(v)\}} dv$$

$$= \int_{c_Q}^{d_Q} \frac{l_Q(v, t)}{\lambda\{\mu^{-1}(v)\}} dv \int_{a_T}^{b_T} k_T\{\mu^{-1}(v), s\} f(s)\, ds$$

$$= \int_{a_T}^{b_T} f(s)\, ds \int_{c_Q}^{d_Q} \frac{l_Q(v, t)\, k_T\{\mu^{-1}(v), s\}}{\lambda\{\mu^{-1}(v)\}} dv, \quad (7)$$

assuming that the order of integration may be inverted.

If the function

$$j(s, t) = \int_{c_Q}^{d_Q} \frac{l_Q(v, t)\, k_T\{\mu^{-1}(v), s\}}{\lambda\{\mu^{-1}(v)\}} dv$$

can be evaluated explicitly, we have the transform pair

$$f(s) = \int_{a_Q}^{b_Q} g(s, t)\, i(t)\, dt, \quad (8)$$

$$i(t) = \int_{a_T}^{b_T} j(s, t) f(s)\, ds. \quad (9)$$

Use of an argument of this sort has been made by A. M. Efross [2], and later by A. Erdélyi, N. W. McLachlan and M. Parodi, in various places, to evaluate Laplace transforms.

2. As an example of the application of the process we consider the integral

$$F(p) = 2^{\nu} \int_0^{\infty} \left(\frac{p}{q}\right)^{\frac{\nu}{2}} K_{\nu}(\sqrt{(pq)})\, g(q)\, dq. \quad (10)$$

Then [1, 5.16 (40)], provided that Rl $q > 0$,

$$f(t) = \tfrac{1}{2} t^{-\nu-1} \int_0^{\infty} e^{-q/4t} g(q)\, dq$$

$$= \tfrac{1}{2} t^{-\nu-1} G\left(\frac{1}{4t}\right), \quad (11)$$

where F and G are the Laplace transforms of f and g. Hence

$$G(s) = 2^{-1-2\nu} s^{-1-\nu} f\left(\frac{1}{4s}\right),$$

and so

$$g(q) = \frac{2^{-1-2\nu}}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} s^{-1-\nu} e^{qs} ds \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} e^{p/4s} F(p) dp.$$

The order of integration may be inverted, and we have

$$g(q) = \frac{2^{-1-\nu}}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\left(\frac{q}{p}\right)^{\frac{1}{2}\nu} I_\nu(\sqrt{(pq)})\ F(p)\,dp, \tag{12}$$

provided that $\mathrm{Rl}\ \nu > -1$, since [1,5·5 (35)]

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{qs+p/4s}s^{-1-\nu}ds = \left(\frac{4q}{p}\right)^{\frac{1}{2}\nu} I_\nu(\sqrt{(pq)}),$$

provided that $\mathrm{Rl}\ \nu > -1$.

We have thus obtained essentially the pair

$$\phi(x) = \int_0^\infty K_\nu(\sqrt{(xy)})\,\psi(y)\,dy, \tag{13a}$$

$$\psi(y) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} I_\nu(\sqrt{(xy)})\phi(x)\,dx, \tag{13b}$$

in which y is a real positive variable, x a complex variable, and $\mathrm{Rl}\ \nu > -1$.

A simple example of this transformation is supplied by the pair

$$\psi(y) = 1, \qquad \phi(x) = \frac{2\Gamma(1+\frac{1}{2}\nu)\,\Gamma(1-\frac{1}{2}\nu)}{x}.$$

3. It is of greater interest to consider an example in which different transformations play a part. A combination of the Laplace and Fourier transformations will produce a pair of formulae due to A. Erdélyi, but unpublished by him [3, p. 189]. The formulae are

$$\phi(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{M_{K,m}(ix)}{i^{m+\frac{1}{2}}\sqrt{x}}\,w(K)\,dK, \tag{14}$$

$$(m, x \text{ real};\ x > 0,\ m > -\tfrac{1}{2})$$

$$w(K) = e^{\frac{1}{2}K\pi}\frac{\Gamma(\frac{1}{2}+m+K)\Gamma(\frac{1}{2}+m-K)}{\{\Gamma(1+2m)\}^2}\int_0^\infty \frac{M_{-K,m}(-iy)}{(-i)^{m+\frac{1}{2}}\sqrt{y}}\,\phi(y)\,dy. \tag{15}$$

They may be derived by the process of section 1 above from the results

$$\mathscr{L}_p\left\{\frac{x^{m-\frac{1}{2}}}{i^{m+\frac{1}{2}}}M_{K,m}(ix)\right\} = \frac{\Gamma(1+2m)}{(p^2+\frac{1}{4})^{\frac{1}{2}+m}}\exp\left(-2iK\tan^{-1}\frac{1}{2p}\right) \tag{16}$$

provided that $m > -\frac{1}{2}$; and

$$\frac{M_{ih,g}(ix)}{(ix)^{g+\frac{1}{2}}} = \frac{\Gamma(1+2g)}{2^{1+2g}\,\Gamma(\frac{1}{2}+g+ih)\,\Gamma(\frac{1}{2}+g-ih)}\int_{-\infty}^{+\infty} e^{\frac{1}{2}x\,\mathrm{th}\,u - ihu}\,\frac{du}{\mathrm{ch}^{1+2g}u}, \quad (17)$$

which holds for real positive x and $\mathrm{Rl}\,g > -\frac{1}{2}$.

Added in proof.

It appears that the transformation described by equations (13) is called the K-transformation and was introduced by C. S. Meijer in 1940. For further information and a table of transforms see Reference 1, Vol II.

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A Matrix Problem Concerning Projections

By Hans Schneider

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The following problem is a slight generalisation of one posed and partly solved by H. Nagler.[1] We shall use A^* for the conjugate transpose of a matrix A. A projection is an idempotent matrix, its latent roots consist of units and zeros.

Problem. Let A be an $n \times m$ matrix with complex elements. Find an $m \times n$ matrix B such that $(I - AB)^*(I - AB)$ is a projection of rank k.

Let the rank of A be r. The rank of AB is less than or equal to r. It is easily proved that the rank of $M = I - AB$ cannot be less than $n - r$. But the rank of M^*M equals that of M, and hence

$$k \geqslant n - r \tag{1}$$

is a necessary condition for the existence of a matrix B with the required property.

We shall next show that (1) is also a sufficient condition. It is enough to find a matrix B such that $I - AB$ is a Hermitian projection of rank k.

The $n \times n$ matrix AA^* is Hermitian of rank r. Hence there exist n Hermitian projections of rank 1 satisfying

$$\sum_1^n E_i = I, \tag{2}$$

$$E_i E_j = 0 \text{ when } i \neq j, \tag{3}$$

$$\sum_1^n \rho_i E_i = AA^*, \tag{4}$$

where the ρ_i are the (non-negative) latent roots of AA^*, supposed arranged so that $\rho_i \neq 0$ for $i = 1, \ldots, r$, and $\rho_i = 0$ for $i = r + 1, \ldots, n$.

Let $$C = \sum_1^{n-k} \sigma_i E_i,$$

where $$\sigma_i = 1/\rho_i \text{ for } i = 1, \ldots, n - k \leqslant r.$$

We note that $$AA^*C = \sum_1^{n-k} E_i.$$

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[1] H. Nagler, "On a certain matrix product with specified latent roots," *Proc. Edinburgh Math. Soc.* (2), 10 (1953), 21-24.

whence it follows that $$I - AA^*C = \sum_{n-k+1}^{n} E_i$$ is a Hermitian projection of rank k. Thus $B = A^*C$ is a matrix having the desired property.

Let $x_1, \ldots, x_n$ form an orthonormal set of latent column vectors of AA^*, where x_i is associated with the latent root ρ_i. It may be remarked that a set of Hermitian projections of rank 1 satisfying (2), (3) and (4) is given by $E_i = x_i x_i^*$ for $i = 1, \ldots, n$.

Suppose now that $k = n - r$. When $m \leqslant n$ and the rank of A is m, then we assert that our method yields $B = (A^*A)^{-1}A^*$, while if $n \leqslant m$ and the rank of A is n, then $B = A^* (AA^*)^{-1}$. The proof of this is left to the reader. In general, it is clear from the method of construction that the solution we have found is not unique.

An $m \times n$ matrix D for which $I - DA$ is a Hermitian projection of rank k can be found in a similar manner, provided that $k \geqq m - r$.

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On Compact Riemannian Manifolds with Zero Ricci Curvature

By T. J. Willmore

(*Received 1st February* 1952.)

1. In this note I prove the following result:—

Theorem. *A compact, orientable, Riemannian manifold M_n, with positive definite metric and zero Ricci curvature, is flat if the first Betti number R_1 exceeds $n-4$.*

In this statement of the theorem it is assumed that the dimensions of M_n are not less than four. If this is not the case, the result is still valid but appears as a purely local result and is true for a metric of arbitrary signature.

To prove the theorem I require two lemmas, the first of which is local in character.

Lemma 1. *A Riemannian space V_n which is a flat extension of a V_3 and has zero Ricci curvature is necessarily flat.*

By a flat extension of a V_r we mean a space whose metric can be written in the form

$$ds^2 = \sum_{\alpha,\beta=1}^{r} g_{\alpha\beta}\, dx^\alpha\, dx^\beta + \sum_{\lambda=r+1}^{n} (dx^\lambda)^2. \tag{1}$$

To prove the lemma we first show that any V_3 of zero Ricci curvature is necessarily flat. This follows from the well-known result [1] that in any V_3 Weyl's conformal curvature tensor C^h_{ijk} vanishes identically, i.e.

$$C^h_{ijk} \equiv R^h_{ijk} + \frac{1}{(n-2)}(\delta^h_j R_{ik} - \delta^h_k R_{ij} + g_{ik}R^h_j - g_{ij}R^h_k) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) = 0. \tag{2}$$

It follows immediately that for any V_3, $R_{ij}=0$ implies $R^h_{ijk}=0$. Now the only non-zero components of the curvature tensor of a space whose metric is of the form (1) arise from the curvature tensor of V_r. It follows that when $r=3$, the relation $R_{ij}=0$ implies $R^h_{ijk}=0$ in any space which is a flat extension of a V_3. This proves Lemma 1.

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[1] L. P. Eisenhart, *Riemannian Geometry* (1926), p. 91.

LEMMA 2. *A compact, orientable, Riemannian manifold M_n with zero Ricci curvature has its first Betti number R_1 equal to the number of linearly independent parallel vector fields defined over M.*

To prove this lemma we use a result discovered independently by Bochner[1] and Lichnerowicz,[2] which in the notation of the latter author states that if $T_{a_1 a_2 \ldots a_p}$ denotes the skew-symmetric tensor associated with a harmonic form of degree p, then

$$\frac{1}{2p}\Delta T^2 = \frac{1}{p} g^{\alpha\beta} T^{a_1 a_2 \ldots a_p}{}_{, a} T_{a_1 a_2 \ldots a_p \beta} - R_{a_1 \beta_1} T^{a_1}{}_{\beta_2 \ldots \beta_p} T^{\beta_1 \beta_2 \ldots \beta_p} + \tfrac{1}{2}(p-1).\, R_{a_1 \beta_1 a_2 \beta_2}.\, T^{a_1 \beta_1 \beta_3 \ldots \beta_p} T^{a_2 \beta_2 \beta_3 \ldots \beta_p},$$

where $$T^2 = T^{a_1 a_2 \ldots a_p} T_{a_1 a_2 \ldots a_p},$$

and Δ denotes the generalised Laplacian operator. If we write $p = 1$ and $R_{a_1 \beta_1} = 0$ this assumes the simpler form

$$\tfrac{1}{2}\Delta T^2 = g^{\alpha\beta} T^{a_1}{}_{, a} T_{a_1, \beta} \tag{3}$$

where the member on the right is certainly non-negative. We now appeal to Bochner's Lemma[1]

If on a compact orientable manifold we have for a given scalar ψ the relation $\Delta\psi \geqq 0$ everywhere, then $\Delta\psi = 0$ everywhere.

From this it follows that harmonic forms of degree one arise only from parallel vector fields defined over the manifold. This completes the proof of Lemma 2.

The proof of the theorem now readily follows; for the relation $R_1 > n - 4$ implies, by Lemma 2, that the space admits at least $n - 3$ parallel vector fields and is thus a flat extension of a V_3. Appeal to Lemma 1 now completes the proof of the theorem.

2. Another result analogous to Lemma 1 is the following, the proof of which is trivial.

LEMMA 3. *A Riemannian space V_n which is a flat extension of a V_4 and has zero* scalar *curvature is necessarily flat.*

In a recent paper[3] Lichnerowicz has proved several interesting results including the following:—

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[1] S. Bochner, *Annals of Math.*, 49 (1948), 379-390.
[2] A. Lichnerowicz, *C. R. Acad. Sci.*, 226 (1948), 1678-80.
[3] A. Lichnerowicz, *C. R. Acad. Sci.*, 230 (1950), 2146-8.

Let K be a compact, orientable manifold with positive definite recurrent metric. Then if the space has zero scalar curvature, it is necessarily symmetric; and if it has zero Ricci curvature it is flat.

Now A. G. Walker has shown[1] that every real recurrent space with positive definite metric is either a V_2, or the flat extension of a V_2. Using this result with Lemma 3 we find not only that these results of Lichnerowicz are purely local in character but also that the weaker assumption of zero scalar curvature implies the stronger conclusion that the space is flat.

3. The question arises to what extent the restrictions in the hypothesis of our theorem can be relaxed. In particular, are the conditions of compactness and positive definite metric sufficient to ensure that a metric with zero Ricci curvature is flat? To disprove this result it would be sufficient to exhibit a compact manifold with the metric

$$ds^2 = x_1^4 (dx_1^2 + dx_2^2 + dx_3^2) + x_1^{-2} dx_4^2,$$

since this metric is positive definite and satisfies the conditions $R_{ij} = 0$, $R^h{}_{ijk} \neq 0$. It is hoped to deal with this question in a later note.

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[1] A. G. Walker, *Proc. London Math. Soc.*, 52 (1950), 36-64.

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A Determinantal Expansion for a Class of Definite Integral

Part 3. Generalised Continued Fractions

By L. R. Shenton

(*Received 2nd March* 1953.)

1. We have shown in [1] that under certain conditions the definite integral $\int_a^b \frac{A^2(x)\,w(x)}{C(x)}\,dx$ may be approximated by a determinantal ratio. It is our object now to develop the theory when $C(x)$ is a polynomial, showing the relation to the continued fraction form for $\int_a^b \frac{w(x)\,dx}{z-x}$. In particular we shall give various forms for the approximants, and an integral form for the numerator.

2. From [1] we have the expansion

$$F(z_1, z_2, \ldots, z_n) = \int_a^b \frac{w(x)\,dx}{\prod_1^n (z_j - x)} = \sum_{s=0}^{\infty} \frac{|\, a_0, \gamma_{01}, \gamma_{12}, \ldots, \gamma_{s-1,s}\,|^2}{\Delta_{s-1}\,\Delta_s} \qquad (1)$$

$$= -\lim_{s\to\infty} \begin{vmatrix} 0 & a_0 & a_1 & \ldots & a_s \\ a_0 & \gamma_{00} & \gamma_{01} & \cdots & \gamma_{0s} \\ a_1 & \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1s} \\ . & . & . & & \\ . & . & . & & \\ . & . & . & & \\ a_s & \gamma_{s0} & \gamma_{s1} & \cdots & \gamma_{ss} \end{vmatrix} \div \Delta_s \qquad (2)$$

$$= \lim_{s\to\infty} \frac{N_{s+1}(z_1, z_2, \ldots, z_n)}{D_{s+1}(z_1, z_2, \ldots, z_n)} \qquad (3)$$

where [1]

$$a_s = \int_a^b \theta_s(z)\,w(x)\,dx, \qquad \gamma_{r,s} = \int_a^b \theta_r(x)\,\theta_s(x) w(x) \prod_1^n (z_j - x)\,dx,$$

$$\Delta_s = D_{s+1}(z) = |\, \gamma_{00}, \gamma_{11}, \ldots \gamma_{ss}\,|,$$

$w(x) \geqq 0$ for $a \leqq x \leqq b$ (a, b finite),

$\int_a^b w(x)\,dx$ exists and is positive,

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[1] $N_s(z_1, z_2, \ldots, z_n)$ and $D_s(z_1, z_2, \ldots z_n)$ etc. will be abbreviated to $N_s(z)$ and $D_s(z)$ when ambiguity is unlikely.

$\prod_1^n (z_j - x) \equiv (z_1 - x)(z_2 - x) \ldots (z_n - x) > 0$ and the z's are distinct,

$\theta_s(x)$ is an arbitrary polynomial of precise degree s with highest coefficient λ_s.

We shall refer to (1) and (2) as continued fractions (C.F.'s) of the nth order, and $R_s(z_1, z_2, \ldots, z_n) = N_s/D_s$ as the sth approximant or convergent.

The expansions (1) and (2) arise from a consideration of the minimum value of

$$S^2 = \int_a^b w(x)\,\Pi(z-x)\left\{\frac{1}{\Pi(z-x)} - \sum_{s=0}^{r-1} A_s q_s(x)\right\}^2 dx \tag{4}$$

where $q_s(x) = |\,\theta_0(x), \gamma_{01}, \gamma_{12}, \ldots, \gamma_{s-1,s}\,|\,/\{(-)^s \lambda_s \Delta_{s-1}\}$, (5)

and $\{q_s(x)\}$ is an orthogonal system with respect to the weight function $w(x)\Pi(z-x)$, the highest coefficient in $q_s(x)$ being unity. Indeed if we write

$$\phi_r = \int_a^b w(x) \prod_1^n (z_j - x)\, q_r^2(x)\, dx \tag{6}$$

then
$$A_r \phi_r = \int_a^b q_r(x) w(x)\, dx \tag{7}$$

and
$$S^2_{\text{Min}} = F(z) - \sum_0^{r-1} A_s^2 \phi_s. \tag{8}$$

It may be remarked in passing that a consideration of the minimum value of $\int_a^b (z-x)\,w(x)\,\{(z-x)^{-1} - \Sigma A_s q_s(x)\}^2 dx$ and of $\int_a^b x(z-x)w(x)\{(z-x)^{-1} - \Sigma A_s^1 q_s^1(x)\}^2\,dx$ leads to continued fractions for $\int_a^b (z-x)^{-1} w(x)\,dx$ related to the 'even' and 'odd' parts of a Stieltjes type of continued fraction. The present approach shows immediately the central part played by orthogonal polynomials, and although in essence both these expressions were considered by Stieltjes [2], it is only at a later stage that the orthogonality property emerges.

3. We shall now consider various forms for $N_s(z)$ and $D_s(z)$. These arise by taking (a) $\theta_s(x) = (\nu - x)^s$;

$$(b)\ \theta_s(x) = p_s(x),$$

where $\{p_s(x)\}$ is an orthonormal system with respect to $w(x)$, and $p_s(x)$ has highest coefficient k_s;

$$(c)\ \theta_s(x) = q_s(x) = \frac{|\ p_s(x),\ p_{s+1}(z_1),\ p_{s+2}(z_2), \ldots,\ p_{s+n}(z_n)\ |}{k_{s+n}\Pi(z-x)\ |\ p_s(z_1),\ p_{s+1}(z_2), \ldots p_{s+n-1}(z_n)\ |} \tag{9}$$

and the system $\{q_s(x)\}$ is orthogonal with respect to $w(x)\Pi(z-x)$.

$$(a)\ \text{Here}\quad \alpha_s = \int_a^b (\nu - x)^s w(x)\,dx = m_s \text{ say}, \tag{10}$$

$$\gamma_{r,s} = \int_a^b (\nu - x)^{r+s} w(x)\Pi(z-x)\,dx = M_{r+s+n} \text{ say}. \tag{11}$$

For particular choices of $w(x)$, m_s is an Appell polynomial.
Further, if $z_1 = z_2 = \ldots = z_n = \nu$ then $\gamma_{r,s} = m_{r+s+n}$.
From (6) we have, in the notation of persymmetric determinants,

$$\phi_r = \frac{P_{r+1}(M_n,\ M_{n+1},\ \ldots,\ M_{n+2r})}{P_r(M_n,\ M_{n+1}, \ldots,\ M_{n+2r-2})} \tag{12}$$

and $\sum_{s=0}^{r-1} A_s^2 \phi_s$

$$= -\begin{vmatrix} 0 & m_0 & m_1 \ldots & m_{r-1} \\ m_0 & M_n & M_{n+1} \ldots & M_{n+r-1} \\ m_1 & M_{n+1} & M_{n+2} \ldots & M_{n+r-2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m_{r-1} & M_{n+r-1} & M_{n+r} \ldots & M_{n+2r-2} \end{vmatrix} \div P_r(M_n,\ M_{n+1}, \ldots M_{n+2r-2}). \tag{13}$$

If the roots z_j are equal and $z_j = \nu$, M_s is to be replaced by m_s.

(b) The polynomials $p_s(x)$ follow a recurrence relation

$$\begin{aligned} &p_s(x) = (A_s x + B_s)\ p_{s-1}(x) - C_s p_{s-2}(x), && s = 2,\ 3,\ \ldots \\ &p_1(x) = (A_1 x + B_1)p_0, && p_0 = k_0 \\ &A_r = k_r/k_{r-1} > 0, && C_r = A_r/A_{r-1} > 0, \end{aligned} \tag{14}$$

which may be written

$$(x - z_1)p_{s-1}(x) = A_s^{-1} p_s(x) - (z_1 + B_s A_s^{-1})p_{s-1}(x) + A_{s-1}^{-1} p_{s-2}(x). \tag{15}$$

We require the following generalisation of (15):

$$\prod_{j=1}^{n}(x - z_j)p_{s-1}(x) = e_{s-1,n}p_{s+n-1} + e_{s-1,n-1}p_{s+n-2} + \ldots + e_{s-1,0}p_{s-1} + e_{s-2,1}p_{s-2} + e_{s-3,2}p_{s-3} + \ldots + e_{s-n-1,n}p_{s-n-1} \tag{16}$$

where $e_{s-1,n} = k_{s-1}/k_{s+n-1}$, and $e_{s,r}$ is to be taken as zero if $s < 0$.
The notation is justified by the identity

$$\int_a^b [p_{s-1}(x)\Pi(x-z)]\,p_r(x)\,w(x)\,dx = \int_a^b [p_r(x)\,\Pi(x-z)]p_{s-1}(x)\,w(x)\,dx.$$

For example,

$$(x - z_1)(x - z_2)\,p_{s-1}(x) = f_{s-1}p_{s+1} + g_{s-1}p_s + h_{s-1}p_{s-1} + g_{s-2}p_{s-2} + f_{s-3}p_{s-3} \quad (s = 1,\ 2,\ \ldots), \tag{17}$$

where
$$\begin{cases} f_{s-1} = A_s^{-1} A_{s+1}^{-1}, \ s = 1, 2, \ldots, \\ -A_s g_{s-1} = B_{s+1} A_{s+1}^{-1} + z_1 + z_2 + B_s A_s^{-1}, \ s = 1, 2, \ldots \\ h_{s-1} = A_s^{-2} + (z_1 + B_s A_s^{-1})(z_2 + B_s A_s^{-1}) + A_{s-1}^{-2}, \ s = 2, 3, \ldots \\ h_0 = A_1^{-2} + (z_1 + B_1 A_1^{-1})(z_2 + B_1 A_1^{-1}). \end{cases}$$

From (6) and (7) we find, after using (16),

$$\phi_r = \frac{(-)^n}{k_r^2} \frac{K_{r+1}(e_{00}, e_{01}, e_{02}, \ldots, e_{0n})}{K_r(e_{00}, e_{01}, e_{02}, \ldots, e_{0n})}, \quad r = 0, 1, \ldots \tag{18}$$

and

$$\sum_{s=0}^{r-1} A_s^2 \phi_s = (-)^n k_0^{-2} \frac{K_{r-1}(e_{10}, e_{11}, \ldots, e_{1n})}{K_r(e_{00}, e_{01}, \ldots, e_{0n})},$$
$$r = 1, 2, \ldots, K_0 = 1, \tag{19}$$

where we have introduced the notation $K_r(e_{00}, e_{01}, \ldots, e_{0n})$ for a generalised continuant determinant of order r, symmetric, with elements $e_{00}, e_{10}, e_{20}, \ldots$ in the diagonal through (1, 1), $e_{01}, e_{11}, e_{21}, \ldots$ in the diagonal through (1, 2), and so on. We shall refer to these as continuants of the nth kind. Thus the ratio of continuants of the 1st kind is related to C.F's of the first order, the determinants concerned consisting of elements in three diagonals only. Similarly C.F.'s of the second order are associated with the ratio of continuants of the 2nd kind which in turn have elements in five diagonals only.

(c) Writing for simplicity

$$| \, p_s(x), p_{s+1}(z_1), \ldots, p_{s+n}(z_n) \, | = A_s(x, z_1, z_2, \ldots, z_n)$$

we have
$$q_s(x) = \frac{A_s(x, z_1, \ldots, z_n)}{k_{s+n} \prod_1^n (z_j - x) A_s(z_1, z_2, \ldots, z_n)}$$

$$= \frac{1}{k_s A_s(z_1, \ldots, z_n)} \sum_{r=0}^{s} | \, p_r(z_1), p_{s+1}(z_2), p_{s+2}(z_3), \ldots, p_{s+n-1}(z_n) \, | \, p_r(x) \tag{20}$$

by a generalisation of a theorem of Darboux quoted in [1], (17).

Hence from (6) and (7) we find

$$\phi_r = \frac{1}{k_r k_{r+n}} \frac{A_{r+1}(z_1, z_2, \ldots, z_n)}{A_r(z_1, z_2, \ldots, z_n)} \tag{21}$$

and

$$\sum_0^{r-1} A_s^2 \phi_s = \sum_{s=0}^{r-1} \frac{k_{s+n} \, | \, z_1^0, p_{s+1}(z_2), p_{s+2}(z_3), \ldots, p_{s+n-1}(z_n) \, |^2}{k_s A_s(z_1, z_2, \ldots, z_n) A_{s+1}(z_1, z_2, \ldots, z_n)}. \tag{22}$$

4. If we now consider the value of $\prod_0^{r-1}\phi_j$, we have from (18) and (21)

$$D_r(z_1, z_2, \ldots, z_n) = (-)^{rn}K_r(e_{00}, e_{01}, \ldots, e_{0n})$$

$$= \frac{A_r(z_1, z_2, \ldots, z_n)}{|\ z_1^0, z_2^1, \ldots, z_n^{n-1}\ |\ \prod_r^{n+r-1} k_j} \tag{23}$$

since $A_0(z_1, \ldots, z_n) = \prod_0^{n-1} k_j\ |\ z_1^0, z_2^1, \ldots, z_n^{n-1}\ |$, and from (12) and (21)

$$D_r(z_1, z_2, \ldots, z_n) = \prod_{j=0}^{r-1} k_j^2\, P_r(M_n, M_{n+1}, \ldots, M_{n+2r-2}) \tag{24}$$

and as a consequence

$$N_r(z_1, z_2, \ldots, z_n) = (-)^{rn}k_0^{-2}K_{r-1}(e_{10}, e_{11}, \ldots, e_{1n})$$

$$= -\prod_{j=0}^{r-1} k_j^2 \begin{vmatrix} 0 & m_0 & m_1 \cdots & m_{r-1} \\ m_0 & M_n & M_{n+1} \cdots & M_{n+r-1} \\ m_1 & M_{n+1} & & \\ \vdots & \vdots & & \\ m_{r-1} & M_{n+r-1} & & M_{n+2r-2} \end{vmatrix} \tag{25}$$

$$= \frac{A_r(z_1, \ldots, z_n)}{\prod_r^{r+n-1} k_j\ |\ z_1^0, z_2^1, \ldots, z_n^{n-1}\ |}$$

$$\times \sum_{s=0}^{r-1} \frac{k_{s+n}\ |\ z_1^0, p_{s+1}(z_2), \ldots, p_{s+n-1}(z_n)\ |^2}{k_s A_s(z_1, \ldots, z_n)\, A_{s+1}(z_1, \ldots, z_n)}, \tag{25a}$$

$$r = 1, 2, \ldots .$$

When the roots z_j are equal, the only change required in (23) is to replace $\dfrac{A_r(z_1, z_2, \ldots, z_n)}{|\ z_1^0, z_2^1, \ldots z_n^{n-1}\ |}$

by $\dfrac{|\ p_r(z_1), p_{r+1}^{(1)}(z_1), p_{r+2}^{(2)}(z_1), \ldots, p_{r+n-1}^{(n-1)}(z_n)\ |}{(n-1)!!}$,

where superscripts refer to derivatives and

$$(n-1)!! = (n-1)!\,(n-2)!\ldots 1!0!.$$

A similar modification is required in (25a).

As an illustration we take $w(x) = 1/\sqrt{(1-x^2)}$, $a = -1$, $b = 1$,

$$\prod_1^n (z_j - x) \equiv (z - x)^n, \qquad z > 1,$$

with $$p_s(x) = \sqrt{\frac{2}{\pi}} \cos s\theta, \qquad \cos\theta = x, \qquad s = 1, 2, \ldots$$

$$p_0(x) = \sqrt{\frac{1}{\pi}} = k_0; \qquad k_s = 2^{s-1}\sqrt{\frac{2}{\pi}}, \qquad s = 1, 2, \ldots$$

and $$m_s = \int_{-1}^{1} \frac{(z-x)^s}{\sqrt{(1-x^2)}}\, dx = \Pi\,(z^2-1)^{s/2} \bar{P}_s\left(\frac{z}{\sqrt{(z^2-1)}}\right)$$

where $\bar{P}_s(x)$ is Legendre's polynomial. With $t = z/\sqrt{(z^2-1)}$ we find from the modified form of (23), and (24),

$$\begin{vmatrix} b_r & b_{r+1} & \ldots & b_{r+n-1} \\ rc_r & (r+1)c_{r+1} & \ldots & (r+n-1)\,c_{r+n-1} \\ r^2b_r & (r+1)^2b_{r+1} & \ldots & (r+n-1)^2 b_{r+n-1} \\ r^3c_r & (r+1)^3c_{r+1} & \ldots & (r+n-1)^3c_{r+n-1} \\ \vdots & \vdots & & \vdots \end{vmatrix}_{(n)} = \frac{2^\chi (n-1)!!}{(t^2-1)^{r(r-1)/2}}\; P_r\{\bar{P}_n, \bar{P}_{n+1}, \ldots, \bar{P}_{n+2r-2}\} \qquad (26)$$

$$n, r = 1, 2, \ldots$$

where $$\begin{cases} b_r = (t+1)^r + (t-1)^r \\ c_r = (t+1)^r - (t-1)^r \\ 2\chi = (n+r)(n+r-1) + (r-1)(r-2) \end{cases}$$

and $\bar{P}_s$ stands for $\bar{P}_s(t)$. The result for $n = 1$ has been given by Geronimus [3].

5. *A formula for the numerators.* Consider the identity

$$|\, x^0, p(z_1), p_{r+1}(z_2), \ldots, p_{r+n-1}(z_n)\,| = \prod_1^n (z_j - x) \sum_{s=0}^{r-1} B_s q_s(x) \qquad (27)$$

where $$B_s\phi_s = \frac{|\, z_1^0, p_{s+1}(z_2), \ldots, p_{s+n-1}(z_n)\,|}{k_s} \frac{A_r(z_1, \ldots, z_n)}{A_s(z_1, \ldots, z_n)}$$

$$s = 0, 1, \ldots, r-1.$$

Then

$$\int_a^b \frac{|\, x^0, p_r(z_1), \ldots, p_{r+n-1}(z_n)\,|}{\prod_1^n (z_j - x) A_r(z_1, \ldots, z_n)}\, w(x)\, dx = \sum_{s=0}^{r-1} \frac{k_{s+n}\,|\, z_1^0, p_{s+1}(z_2), \ldots, p_{s+n-1}(z_n)\,|^2}{k_s\, A_s(z_1, \ldots, z_n)\, A_{s+1}(z_1, \ldots, z_n)}.$$

Hence from (22) it follows that

$$N_r(z_1, z_2, \ldots, z_n) = \int_a^b \frac{|\, x^0, p_r(z_1), p_{r+1}(z_2), \ldots, p_{r+n-1}(z_n)\,|}{\prod_r^{r+n-1} k_j\,|\, x^0, z_1^1, z_2^2, \ldots, z_n^n\,|}\, w(x)dx \qquad (28)$$

since $\prod_1^n (z_j - x) \mid z_1^0, z_2^1, \ldots, z_n^{n-1} \mid = \mid x^0, z_1^1, z_2^2, \ldots, z_n^n \mid$.

When $z_1 = z_2 = \ldots = z_n = z$, the determinant in the numerator of (28) is to be replaced by $\mid x^0, p_r(z), p_{r+1}^{(1)}(z), \ldots, p_{r+n-1}^{(n-1)}(z) \mid$, and that in the denominator by $(z - x)^n(n - 1)!!$.

In the particular case $n = 1$ we have the well-known formula

$$N_r(z) = k_r^{-1} \int_a^b \frac{p_r(z) - p_r(x)}{z - x} w(x)\, dx,$$

and from (23) $D_r(z) = k_r^{-1} p_r(z)$, the ratio of these being the rth convergent of the C.F.

$$\frac{A_1 k_0^{-2}}{A_1 z + B_1} - \frac{C_2}{A_2 z + B_2} - \frac{C_3}{A_3 z + B_3} - \ldots$$

We shall consider the recurrence relations for the numerators and denominators of generalised C.F.'s, and some special properties of second order C.F.'s in Part 4.

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On an Asymptotic Integral

By L. C. Hsu

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1. This note gives an asymptotic evaluation of an integral of the form

$$I_n = \int_a^b \{f_n(x)\}^n \, g(x) \, dx, \tag{1}$$

as n tends to infinity, where $\{f_n(x)\}$ is a sequence of real-valued functions. The theorem to be established is a natural extension of B. Levi's generalised Laplace-Darboux theorem (1, 341-51); it gives a rule for evaluating a wider class of asymptotic integrals.

In what follows $(x, n) \to (\xi, \infty)$ denotes that x, n tend to ξ, ∞ respectively and independently. For instance, $\phi(x, n) \to 0$ as $(x, n) \to (\xi, \infty)$ means that for any given positive ϵ there exist positive numbers δ and N such that $| \phi(x, n) | < \epsilon$ whenever $| x - \xi | < \delta$ and $n > N$.

Theorem. *Let* $f_n(x)$ $[n = 1, 2, 3, \ldots]$ *and* $g(x)$ *be real functions such that*

(A) $g(x)$ *and each* $f_n(x)$ *are integrable over* (a, b);

(B) *there is a number* A, *independent of* x *and* n, *such that* $| f_n(x) | < A$, $| g(x) | < A$;

(C) $f_n(x)$ *attains a positive absolute maximum at* $x = \xi_n$;

(D) *for each positive number* d, *there is a positive number* δ *(depending on* d *but not on* n*) such that* $| f_n(x) | \leqq f_n(\xi_n) - \delta$ *whenever* $| x - \xi_n | \geqq d$ *and* $a \leqq x \leqq b$;

(E) *as* n *tends to infinity,* ξ_n *tends to* ξ, *where* $a < \xi < b$;

(F) $g(x)$ *is continuous at* $x = \xi$ *with* $g(\xi) \neq 0$;

(G) *there are positive constants* h *and* k *such that*

$$\lim_{(x, n) \to (\xi, \infty)} | f_n(x) - f_n(\xi_n) | / | x - \xi_n |^h = k. \tag{2}$$

Then for n *large*

$$I_n \sim (2/h) \, \Gamma(1/h) \, \{f_n(\xi_n)\}^n \, g(\xi) \, \{f_n(\xi_n)/nk\}^{1/h}. \tag{3}$$

2. The essential step in the proof is to determine a suitable small interval containing the variable point $x = \xi_n$ and then let n tend to infinity in order to get the dominant asymptotic value. Write

$$f_n(x) - f_n(\xi_n) = -k \mid x - \xi_n \mid^h \{1 + R(x, \xi_n)\} \tag{4}$$

so that by condition (G), $R(x, \xi_n) \to 0$ as $(x, n) \to (\xi, \infty)$. We now have

$$\begin{aligned} \log f_n(x) &= \log f_n(\xi_n) + \log \{1 - k \mid x - \xi_n \mid^h \{f_n(\xi_n)\}^{-1}(1 + R)\} \\ &= \log f_n(\xi_n) - k \mid x - \xi_n \mid^h \{f_n(\xi_n)\}^{-1}(1 + R + R'), \end{aligned} \tag{5}$$

say,[1] where $R' = O(\mid x - \xi_n \mid^h)$ so that $R' \to 0$ as $(x, n) \to (\xi, \infty)$. Denote $R + R'$ by $\theta(x, \xi_n)$. Then, given any small positive nnmber ϵ, there are positive numbers Δ and M such that $\mid \theta \mid < \epsilon$ whenever $\mid x - \xi \mid < 2\Delta$ and $n > M$. We may take M so large that $\mid \xi - \xi_n \mid < \Delta$ whenever $n > M$. Thus $\mid x - \xi_n \mid < \Delta$ implies $\mid x - \xi_n \mid < 2\Delta$. We may also assume $a \leqq \xi_n - \Delta$, $\xi_n + \Delta \leqq b$ for $n > M$.

If we now use (5), if we make the change of variable

$$nk\{f_n(\xi_n)\}^{-1} \mid x - \xi_n \mid^h = t, \tag{6}$$

and if we denote $\theta(x, \xi_n)$ by $\theta_1(t)$ for $\xi_n \leqq x \leqq \xi_n + \Delta$ and by $\theta_2(t)$, for $\xi_n - \Delta \leqq x \leqq \xi_n$, we obtain

$$\begin{aligned} J_n &\equiv \int_{\xi_n - \Delta}^{\xi_n + \Delta} \left\{\frac{f_n(x)}{f_n(\xi_n)}\right\}^n \left\{\frac{nk}{f_n(\xi_n)}\right\}^{1/h} dx \\ &= \left\{\frac{nk}{f_n(\xi_n)}\right\}^{1/h} \int_{\xi_n - \Delta}^{\xi_n + \Delta} \exp\left[-nk(1 + \theta)\{f^n(\xi_n)\}^{-1} \mid x - \xi_n \mid^h\right] dx \\ &= \frac{1}{h}\int_0^T \left[e^{-t[1 + \theta_1(t)]} + e^{-t[1 + \theta_2(t)]}\right] t^{1/h - 1} dt, \end{aligned} \tag{7}$$

where $T = nk\{f_n(\xi_n)\}^{-1}\Delta^h$.

Now suppose that ϵ and Δ are fixed. Then T, which is not less than $nk\Delta^h/R$, tends to infinity with n. Since $\mid \theta_1 \mid < \epsilon$ and $\mid \theta_2 \mid < \epsilon$, we see from (7) that

$$\frac{2}{h}\left(\frac{1}{1 + \epsilon}\right)^{1/h} \int_0^{T(1 + \epsilon)} e^{-u} u^{1/h - 1} du \leqq J_n \leqq \frac{2}{h}\left(\frac{1}{1 - \epsilon}\right)^{1/h} \int_0^{T(1 - \epsilon)} e^{-u} u^{1/h - 1} du.$$

If we now let n tend to infinity, we find that

$$\frac{2}{h}\left(\frac{1}{1 + \epsilon}\right)^{1/h} \Gamma\left(\frac{1}{h}\right) \leqq \underline{\lim}\, J_n \leqq \overline{\lim}\, J_n \leqq \frac{2}{h}\left(\frac{1}{1 - \epsilon}\right)^{1/h} \Gamma\left(\frac{1}{h}\right). \tag{8}$$

[1] Here we use the fact that $f_n(\xi_n)$ cannot tend to zero as n tends to infinity. This follows from condition (D).

Let us now consider the integral

$$J_n^* \equiv \int_{\xi_n - \Delta}^{\xi_n + \Delta} \left\{\frac{f_n(x)}{f_n(\xi_n)}\right\}^n \left\{\frac{nk}{f_n(\xi_n)}\right\}^{1/h} g(x)\,dx. \tag{9}$$

Since $g(x)$ is continuous at $x = \xi$, we may assume that Δ is chosen so that [1]

$$g(\xi)(1-\epsilon) \leqq g(x) \leqq g(\xi)(1+\epsilon)$$

whenever $|x - \xi_n| < \Delta$. Thus from (8) and (9) we may infer that

$$\frac{2(1-\epsilon)}{h(1+\epsilon)^{1/h}}\Gamma\left(\frac{1}{h}\right)g(\xi) \leqq \lim J_n^* \leqq \overline{\lim}\, J_n^* \leqq \frac{2(1+\epsilon)}{h(1-\epsilon)^{1/h}}\Gamma\left(\frac{1}{h}\right)g(\xi). \tag{10}$$

On the other hand, by hypothesis (D), there is a positive number δ (independent of n) such that $|f_n(x)| \leqq f_n(\xi_n) - \delta$ whenever $|x - \xi_n| \geqq \Delta$ and $a \leqq x \leqq b$. Hence for these values of x

$$\left|\frac{f_n(x)}{f_n(\xi_n)}\right| \leqq \left|\frac{f_n(\xi_n)-\delta}{f_n(\xi_n)}\right| \leqq \left|1 - \frac{\delta}{A}\right| = \rho$$

say, where $0 < \rho < 1$. It now follows that, with Δ fixed,

$$J_n^{**} = \left(\int_a^{\xi_n - \Delta} + \int_{\xi_n + \Delta}^b\right)\left\{\frac{f_n(x)}{f_n(\xi_n)}\right\}^n \left\{\frac{nk}{f_n(\xi_n)}\right\}^{1/h} g(x)dx = O(\rho^n n^{1/h}) \to 0$$

as n tends to infinity. We may therefore replace J_n^* by $(J_n^* + J_n^{**})$ in equation (10). Since ϵ is arbitrary, and since $(J_n^* + J_n^{**})$ does not depend on ϵ, it now follows that

$$\lim_{n \to \infty}\left(J_n^* + J_n^{**}\right) = \frac{2}{h}\Gamma\left(\frac{1}{h}\right)g(\xi).$$

This is equivalent to (3), and our theorem is established.

3. Concrete examples are easily found for illustrating the use of the formula (3). A simple example is, for $n \to \infty$,

$$n^{1/s}\int_{-1/2}^{1/2}\left(1 + \frac{1}{n} - \left|x - \frac{1}{\sqrt{n}}\right|^s\right)^n \sin^{-1}(1 - |x|)\,dx \sim (1/s)\Gamma(1/s)\pi e,$$

where $s > 0$ and $0 \leqq \sin^{-1} y \leqq \frac{\pi}{2}$ $(0 \leqq y \leqq 1)$. As consequences of our theorem we now mention two important cases as follows:

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[1] There are slight changes here if $g(\xi)$ is negative; $g(\xi)$ is not zero by hypothesis (F).

I. *Levi's case.* If $f_n(x) = f(x)$, then $\xi_n = \xi$ and the equation (2) becomes

$$\lim_{x \to \xi} | f(x) - f(\xi) | / | x - \xi |^h = k. \tag{2}'$$

In this case we have

$$\int_a^b \Big(f(x)\Big)^n g(x)dx \sim \frac{2}{h}\,\Gamma\left(\frac{1}{h}\right)\Big(f(\xi)\Big)^n g(\xi)\left(\frac{f(\xi)}{nk}\right)^{1/h}. \tag{3}'$$

II. *Laplace-Darboux case.* In the case of Levi, if $f(x)$ is continuous together with its derivatives $f'(x)$, $f''(x)$ so that $f'(\xi) = 0$, $f''(\xi) < 0$, and (2)′ is true for $h = 2$, $k = -\frac{1}{2}f''(\xi)$, then (3)′ reduces to the classical asymptotic formula of Laplace and Darboux ([2], [3], [4]) :

$$\int_a^b \Big(f(x)\Big)^n g(x)\,dx \sim \Big(f(\xi)\Big)^{n+1/2} g(\xi)\left(\frac{-2\pi}{nf''(\xi)}\right)^{1/2}. \tag{3}''$$

4. Two remarks are worthy of mention. (i) In general the constants h and k may always be determined by means of the following equation

$$\lim_{n \to \infty} \lim_{x \to \xi_n} | f_n(x) - f_n(\xi_n) | / | x - \xi_n |^h = k. \tag{2}^*$$

For it is easily seen that (2)* is implied by (2), in view of (4). In the case when ξ_n is a constant, (2)* and (2)′ are equivalent. (ii) If our hypothesis (F) is replaced by

$(F)^*$ *$g(x) \epsilon L(a, b)$ and $g(x)$ possesses limits on both sides of $x = \xi_n$* $(n = 1, 2, 3, \ldots)$,

then by almost the same treatment as used before we easily obtain

$$I_n \sim (1/h)\,(g(\xi_n -) + g(\xi_n +))\,(f_n(\xi_n))^n (f_n(\xi_n)/nk)^{1/h}\,\Gamma(1/h). \tag{3}^*$$

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